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Analytical Approximation for the Price Dynamics of Spark Spread Options

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Abstract

This paper presents an analytic approximation for the pricing dynamics of spark spread options in terms of Fourier transforms. We propose to model the spark spread, that is, the price difference of electricity and gas, directly using a mean-reverting model with diffusion and jumps. The model is analyzed empirically, and shown to fit observed data in the UK reasonably well. The main advantage with the model is that the spark spread of electricity and gas forwards, being forwards with delivery over periods, can be priced analytically. The price dynamics for different spark spread options with electricity and gas forwards as underlying, is analytically derived through Fourier transforms. These pricing expressions allow for efficient numerical valuations via the fast Fourier transform technique.

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1 Introduction

Spark spread options are option contracts written on the price difference of electricity and gas, being e.g., the spot or forward price. Such contracts can provide a producer of electricity, say, protection against too high gas prices, or be used as a tool to value a power plant (as a real option). The spark spread also appears in exotic derivatives contracts like tolling agreements, where the holder is allowed to produce and sell electricity at advantageous prices, or simply hold back if the price differential is less than a given threshold. Tolling agreements are typical examples of swing options, where the holder has additional optionality in choosing the level and timing for striking the contract.

It is well-known, at least in the case when the two underlying processes are geometric Brownian motions, that there are no analytical pricing formula like the Black & Scholes formula for spark spread options, except in the case where the strike is equal to zero. When the option is a call or put on the price differential directly, then one may use the Margrabe Formula (see Margrabe (1978)) to value the option. The most common models for spot energies are of an exponential Ornstein-Uhlenbeck type (known as the Schwartz model, see Schwartz (1997)), and include features like mean-reversion and jumps. These processes generalize the geometric Brownian motion dynamics, and just add up the complexity of the pricing problem for spread options.

Most commonly, the spark spread options are written on the difference of the forward prices of the two underlying commodities. Forwards in gas and electricity are traded with delivery over a *period* rather than at a fixed future point in time. If we use a Schwartz model for both energies, it is possible to derive explicit dynamics for fixed-delivery forwards (see e.g. Benth et al. (2003)), but for forward contracts delivering over a time period there exist no explicit dynamics. Thus, to price spark spread options when the spot price processes are modelled marginally by exponential processes, we are forced to apply numerical valuation techniques for price path simulation and numerical integration. The standard approaches are Monte Carlo simulation or numerical solution of multi-dimensional partial differential equations.

Insisting on modeling the price processes of the two underlyings using geometric Brownian motion, and considering spark spread options on fixed-delivery forward contracts, lead us to calculating the expected value of a payoff function depending on the difference between two lognormal random variables. In the paper by Carmona and Durrleman (2003) it is suggested to approximate the difference of two lognormal random variables with a normal random variable. Lima (2005) observed that the difference between two ex-

ponential Ornstein-Uhlenbeck processes could not be approximated very well with a normal distribution, but instead showed features like skewness and heavy tails. This led Lima (2005) to propose to model the difference using the Normal inverse Gaussian distribution (NIG) introduced in finance by Barndorff-Nielsen (1998).

Taking the ideas above one step further, one may ask whether the spark spread *data* can be modelled *directly* using a mean-reverting process with jumps, having additive noise rather than geometric. This paper proposes a seasonally varying Ornstein-Uhlenbeck process to model the spark spread data, where the random innovations are given by a Wiener process and compound Poisson processes. We let the small price difference variations be modelled by a diffusion, while the bigger jumps that we observe in the market are captured by a compound Poisson process. Analysing gas and electricity data from the UK demonstrates that this additive model fits the observed spark spreads reasonably well.

The main advantage with this model is the pricing tractability. We are able to derive an explicit dynamics of the spark spread forwards for arbitrary delivery periods, and the price dynamics of general option contracts written on the spark spread of electricity and gas forwards can be represented by explicit inverse Fourier transforms. The pricing formula when we include jumps in the modeling is not as explicit as the Margrabe formula, say, but can easily be calculated using numerical methods like the fast Fourier transform. To base the pricing on a two-dimensional stochastic process, one must use advanced numerical tools which will lead to more complicated and less efficient methods for finding the price. Moreover, if one wants to analyse the price dynamics, one is forced to numerically simulate conditional expectations in the case of a Monte Carlo approach, which needs a large number of simulations before converging. Our simple formula is more efficient in finding the price dynamics, and we claim that it is better if one wants to perform risk evaluations on a portfolio, say, consisting of several spark spread option contracts and other nonlinear products.

The disadvantage with our way to analyse the problem is that we lose the connection with the marginal processes in the pricing formula (except indirectly via the estimated parameters). To find a two-dimensional stochastic process that marginally models gas and electricity, and also includes the correct dependency structure may be a challenging task. It is known that energies must be modelled by jump processes, however, it is not straightforward to build reasonable models where the dependency comes into play in an empirically correct way. Copulas is a modeling tool that seems to be promising, however,

we shall not pursue this idea further in the present paper, but leave it for future research (see Benth and Kettler (2006)). We remark that such a modeling does not lead to any simple pricing formula for spark spread options, on the contrary, they need to be calculated by numerical techniques as discussed above.

The paper is organized as follows: In Section 2 we define the market and introduce the spark spread options that we are going to analyse, together with some discussions about our proposed model. In Section 3 we continue with defining the stochastic dynamics of the spark spread, along with the characteristic functions required for calculating the forward price dynamics. Using Fourier methods, we present formulas for spark spread options price dynamics in Section 4, being in a suitable form for applying fast Fourier transform techniques. In Section 5 we perform an empirical study of electricity and gas data collected in the UK, and demonstrate the reasonability of our model together with a discussion of option pricing. We conclude in Section 6.

2 Spark spread options

In this section we introduce at a formal level the spark spread options that we are going to analyse, and establish some connections related to our proposed modeling view. Let $E(t)$ be the electricity spot price and $G(t)$ the gas spot price at time t , modelled as two stochastic processes on the complete probability space (Ω, \mathcal{F}, P) equipped with a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. T is the time horizon of the market. If c is the heat rate, that is, the factor converting gas prices into the units of electricity prices, the spark spread difference between electricity and gas is

$$S(t) = E(t) - cG(t). \quad (2.1)$$

We call $S(t)$ the *spark spread*. A forward contract on electricity and gas deliver over a period rather than at a fixed point in time. We consider a forward contract written on a spot with price $X(t)$ having delivery over the period $[\tau_1, \tau_2]$. The forward price at time $t \leq \tau_1$ is defined to be the risk-neutral expectation of the delivery, that is;

$$F(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} X(u) du \mid \mathcal{F}_t \right]. \quad (2.2)$$

This forward price can be derived using the theory of no-arbitrage together with the fact that it is costless to enter a forward contract. In the market, such

contracts are settled either physically or financially. Let F_{el} and F_{gas} denote the electricity and gas forward, respectively.

Consider a spark spread option written on the difference of electricity and gas. The price of an option with exercise at time T with strike K is defined as

$$C(t) = e^{-r(T-t)} \mathbb{E}_Q [\max (F_{\text{el}}(T, \tau_1, \tau_2) - cF_{\text{gas}}(T, \tau_1, \tau_2) - K, 0) | \mathcal{F}_t] , \quad (2.3)$$

where r is the risk-free interest rate. The risk-neutral probability Q is the same for both forwards and the option. It is a probability measure that will change the underlying characteristics of the processes. We suppose that there is a measure Q describing the risk preferences in both the gas and electricity markets. This can be done without loss of generality since we model both markets jointly, and the Q -probability refer to the stochastic processes.

Using the definitions of $F_{\text{el}}(t, \tau_1, \tau_2)$ and $F_{\text{gas}}(t, \tau_1, \tau_2)$, we find

$$\begin{aligned} F_{\text{spread}}(t, \tau_1, \tau_2) &= F_{\text{el}}(t, \tau_1, \tau_2) - cF_{\text{gas}}(t, \tau_1, \tau_2) \\ &= \frac{1}{\tau_2 - \tau_1} \mathbb{E}_Q \left[\int_{\tau_1}^{\tau_2} E(u) - cG(u) du | \mathcal{F}_t \right] \\ &= \mathbb{E}_Q \left[\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} S(u) du | \mathcal{F}_t \right] . \end{aligned} \quad (2.4)$$

Thus, we see that the spark spread option can be considered as an option written on a forward contract with delivery of the spread difference over the period $[\tau_1, \tau_2]$. Motivated from this, we can price the spread option based on a model for the spread difference directly, rather than modeling the electricity and gas separately.

Note that this view is not far from the idea introduced in Carmona and Durrleman (2003). There the authors model each commodity as a geometric model, linked by a correlated two-dimensional Brownian motion. Only for $K = 0$ it is possible to price analytically in terms of a Black & Scholes type formula. To deal with a strike which is different than zero, they argue that the difference between two lognormal variables can be approximated by a normal distribution, for which prices can be explicitly found. In Lima (2005) it is demonstrated that the difference for electricity and gas is not very well captured by the normal distribution when basing the marginal dynamics on a mean-reversion model. The joint geometric model gives heavy tails in the difference, something that is important for spread options which are far out of the money. We argue further that when applying geometric models marginally, we lose in general the analytical expression for the forward price dynamics. The approach of modeling the spark spread directly by an additive model, provides

us with an explicit dynamics for the spark spread forward and thus tractable expressions in terms of Fourier transforms for the option prices. The reader is referred to Benth, Kallsen and Meyer-Brandis (2005) for additive models applied to electricity markets. We now move on with a more detailed analysis.

3 Modeling and analysis of the spark spread

In this Section we propose a model for the spark spread. Moreover, we derive some key formulas involving the characteristic function of different expressions of the spark spread, that becomes useful when pricing forward contracts and options on these.

We model the spark spread by a non-Gaussian Ornstein-Uhlenbeck process: Suppose the dynamics of $S(t)$ is decomposed into a seasonal part $\Lambda(t)$ and a stochastic part $\tilde{S}(t)$ as follows,

$$S(t) = \Lambda(t) + \tilde{S}(t). \quad (3.1)$$

The dynamics of \tilde{S} is given by a non-Gaussian Ornstein-Uhlenbeck process,

$$d\tilde{S}(t) = -\alpha\tilde{S}(t) dt + \sigma dB(t) + dL(t), \quad (3.2)$$

where B is a Brownian motion and L is a pure-jump Lévy process. The volatility σ is a non-negative constant. We denote by $\psi(\theta)$ the cumulant function of L , defined as (θ being a real number)

$$\psi(\theta) = \ln \mathbb{E} [\exp (i\theta L(1))] . \quad (3.3)$$

The Lévy-Kintchine Formula gives an explicit expression for ψ in terms of the Lévy measure of L ,

$$\psi(\theta) = \int_{\mathbb{R}} \{e^{i\theta z} - 1 - i\theta z 1_{|z|<1}\} \ell(dz). \quad (3.4)$$

Here, the Lévy measure $\ell(dz)$ is a σ -finite Borel measure on the real line with $\ell(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(1, z^2) \ell(dz) < \infty .$$

We suppose that L has finite variance, which is equivalent to assuming that

$$\int_{\mathbb{R}} z^2 \ell(dz) < \infty . \quad (3.5)$$

We note that for a measurable and bounded function g , the characteristic function of $\int_a^b g(u) dL(u)$, $0 \leq a < b$ is

$$\mathbb{E} \left[\exp \left(i\theta \int_a^b g(u) dL(u) \right) \right] = \exp \left(\int_a^b \psi(\theta g(u)) du \right). \quad (3.6)$$

A useful observation for later is the relation

$$\mathbb{E} \left[\int_a^b g(u) dL(u) \right] = (-i\psi'(0)) \int_a^b g(u) du. \quad (3.7)$$

For more theory on Lévy processes and their applications in finance, see Shiryaev (1999).

We need to specify the dynamics of \tilde{S} under a risk-neutral probability Q , and we suppose from now on that Q is defined via the Girsanov transform for which

$$dW(t) = dB(t) - \frac{\mu_{mpr}}{\sigma} dt \quad (3.8)$$

is a Brownian motion. Here, μ_{mpr} is a parameter that we will interpret as the market price of risk, measuring the reward that market traders charge for the risk of not being able to hedge perfectly any derivative using gas and/or electricity. The market price of risk is an additional parameter in the model specification that needs to be calibrated from historical derivatives price data. Note that under this change of probability, the characteristics of L remains unaltered, which in other words mean that we suppose that there is no market price of risk connected to the jumps. This is a frequently used assumption (see e.g. Cartea and Figueroa (2005)), simplifying the further analysis considerably. The Q -dynamics of \tilde{S} becomes

$$d\tilde{S}(t) = \left(\mu_{mpr} - \alpha \tilde{S}(t) \right) dt + \sigma dW(t) + dL(t). \quad (3.9)$$

We have the following explicit results for $\tilde{S}(t)$ and its time integral:

Lemma 3.1. *Under the risk-neutral measure Q , we have*

$$\tilde{S}(t) = \tilde{S}(0)e^{-\alpha t} + \frac{\mu_{mpr}}{\alpha}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha u} dW(u) + e^{-\alpha t} \int_0^t e^{\alpha u} dL(u), \quad (3.10)$$

and

$$\int_{\tau_1}^{\tau_2} \tilde{S}(u) du = -\frac{1}{\alpha} \tilde{S}(\tau_2) + \frac{1}{\alpha} \tilde{S}(\tau_1) + \frac{\mu_{mpr}}{\alpha}(\tau_2 - \tau_1) + \frac{\sigma}{\alpha} (W(\tau_2) - W(\tau_1)) \quad (3.11)$$

$$+ \frac{1}{\alpha} (L(\tau_2) - L(\tau_1)) .$$

Proof. Equation (3.11) follows by a direct integration of the stochastic differential equation (3.2). Appealing to the Itô Formula of semimartingales we find (3.10). \square

From these two explicit representations, we can derive the characteristic functions of $\tilde{S}(t)$ and $\int_{\tau_1}^{\tau_2} \tilde{S}(t) dt$, being key formulas in calculating forward prices and options on these. The characteristic functions are stated in next two Lemmas:

Lemma 3.2. *For $\theta \in \mathbb{R}$ and $0 \leq t \leq \tau$, it holds that*

$$\begin{aligned} \mathbb{E}_Q \left[\exp \left(i\theta \tilde{S}(\tau) \right) \mid \mathcal{F}_t \right] &= \exp \left(i\theta (\mu(t, \tau) + e^{-\alpha(\tau-t)} \tilde{S}(t)) \right. \\ &\quad \left. - \frac{\theta^2 \sigma^2}{4\alpha} (1 - e^{-2\alpha(\tau-t)}) + \Psi(t, \tau; \theta) \right), \end{aligned} \quad (3.12)$$

where

$$\begin{aligned} \mu(t, \tau) &= \frac{\mu_{mpr}}{\alpha} \{1 - e^{-\alpha(\tau-t)}\} \\ \Psi(t, \tau; \theta) &= \int_t^\tau \psi(\theta e^{-\alpha(\tau-u)}) du . \end{aligned}$$

Proof. For notational simplicity, define $\tilde{L} = \sigma W + L$, which is a Lévy process. Integrating (3.2) from t to τ yields

$$\tilde{S}(\tau) = \tilde{S}(t)e^{-\alpha(\tau-t)} + \frac{\mu_{mpr}}{\alpha} (1 - e^{-\alpha(\tau-t)}) + e^{-\alpha\tau} \int_t^\tau e^{\alpha u} d\tilde{L}(u). \quad (3.13)$$

From the \mathcal{F}_t -measurability of $\tilde{S}(t)$ and independence property of the increments of \tilde{L} , we find

$$\begin{aligned} \mathbb{E}_Q \left[\exp \left(i\theta \tilde{S}(\tau) \right) \mid \mathcal{F}_t \right] &= \exp \left(i\theta \mu(t, \tau) + i\theta e^{-\alpha(\tau-t)} \tilde{S}(t) \right) \\ &\quad \times \mathbb{E}_Q \left[e^{i\theta e^{-\alpha\tau} \int_t^\tau e^{\alpha u} d\tilde{L}(u)} \right]. \end{aligned}$$

Hence, the Lemma follows after appealing to the characteristic function of the integral of the Lévy process \tilde{L} , as given in (3.6). \square

The next Lemma states the similar result for $\int_{\tau_1}^{\tau_2} \tilde{S}(u) du$:

Lemma 3.3. For $\theta \in \mathbb{R}$ and $0 \leq t \leq \tau_1 < \tau_2$, it holds that

$$\begin{aligned} \mathbb{E}_Q \left[\exp \left(i\theta \int_{\tau_1}^{\tau_2} \tilde{S}(u) du \right) \mid \mathcal{F}_t \right] &= \exp \left(i\theta \lambda_1(t, \tau_1, \tau_2) \right) \\ &+ \Psi(t, \tau_1, \tau_2; \theta) + i\theta \lambda_2(t, \tau_1, \tau_2) \tilde{S}(t) \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} \lambda_1(t, \tau_1, \tau_2) &= \frac{\mu_{mpr}}{\alpha} \left\{ (\tau_2 - \tau_1) - \frac{1}{\alpha} (e^{-\alpha(\tau_1-t)} - e^{-\alpha(\tau_2-t)}) \right\} \\ \lambda_2(t, \tau_1, \tau_2) &= \frac{1}{\alpha} \{ e^{-\alpha(\tau_1-t)} - e^{-\alpha(\tau_2-t)} \} \\ \Psi(t, \tau_1, \tau_2; \theta) &= \int_t^{\tau_1} \psi \left(\frac{\theta}{\alpha} \{ e^{-\alpha(\tau_1-u)} - e^{-\alpha(\tau_2-u)} \} \right) \\ &\quad - \frac{\theta^2 \sigma^2}{2\alpha^2} \{ e^{-\alpha(\tau_1-u)} - e^{-\alpha(\tau_2-u)} \}^2 du \\ &\quad + \int_{\tau_1}^{\tau_2} \psi \left(\frac{\theta}{\alpha} \{ 1 - e^{-\alpha(\tau_2-u)} \} \right) - \frac{\theta^2 \sigma^2}{2\alpha^2} \{ 1 - e^{-\alpha(\tau_2-u)} \}^2 du. \end{aligned}$$

Proof. The proof follows the same lines as in Lemma 3.2: Use $\tilde{L} = \sigma W + L$, and observe from (3.13) that

$$\begin{aligned} \tilde{S}(\tau_1) - \tilde{S}(\tau_2) &= \tilde{S}(t) \{ e^{-\alpha(\tau_1-t)} - e^{-\alpha(\tau_2-t)} \} - \frac{\mu_{mpr}}{\alpha} \{ e^{-\alpha(\tau_1-t)} - e^{-\alpha(\tau_2-t)} \} \\ &\quad + e^{-\alpha\tau_1} \int_t^{\tau_1} e^{\alpha u} d\tilde{L}(u) - e^{-\alpha\tau_2} \int_t^{\tau_2} e^{\alpha u} d\tilde{L}(u). \end{aligned}$$

Hence, from (3.11), the \mathcal{F}_t -measurability of $\tilde{S}(t)$ and the independent increment property of \tilde{L} , we get that

$$\begin{aligned} \mathbb{E}_Q \left[\exp \left(i\theta \int_{\tau_1}^{\tau_2} \tilde{S}(u) du \right) \mid \mathcal{F}_t \right] &= \exp \left(i\theta \lambda_1(t, \tau_1, \tau_2) + i\theta \lambda_2(t, \tau_1, \tau_2) \tilde{S}(t) \right) \\ &\quad \times \mathbb{E}_Q \left[\exp \left(i \frac{\theta}{\alpha} \int_{\tau_1}^{\tau_2} 1 - e^{-\alpha(\tau_2-t-u)} d\tilde{L}(u) \right) \right] \\ &\quad \times \mathbb{E}_Q \left[\exp \left(i \frac{\theta}{\alpha} (e^{-\alpha(\tau_1-t)} - e^{-\alpha(\tau_2-t)}) \int_t^{\tau_1} e^{\alpha u} d\tilde{L}(u) \right) \right]. \end{aligned}$$

The Lemma follows from (3.6). □

We are now in the position to derive the dynamics for the forward price difference underlying the spark spread option:

Proposition 3.4. *For $0 \leq t \leq \tau_1 < \tau_2$, we have that*

$$F_{\text{spread}}(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du + \lambda(t, \tau_1, \tau_2) + \delta(t, \tau_1, \tau_2) (S(t) - \Lambda(t)) , \quad (3.15)$$

where

$$\lambda(t, \tau_1, \tau_2) = \frac{\mu_{\text{mpr}} - i\psi'(0)}{\alpha} (1 - \delta(t, \tau_1, \tau_2)) \quad (3.16)$$

$$\delta(t, \tau_1, \tau_2) = \frac{1}{\alpha(\tau_2 - \tau_1)} (e^{-\alpha(\tau_1 - t)} - e^{-\alpha(\tau_2 - t)}) . \quad (3.17)$$

Proof. Differentiating the right-hand side of the key formula in Lemma 3.3 and letting $\theta = 0$ lead to the desired result after observing that $\Psi(t, \tau_1, \tau_2; 0) = 0$. \square

We remark that applying exponential Ornstein-Uhlenbeck models for the spot prices will not lead to any explicit dynamics for the forward, but has to be represented as an integral. This makes it impossible to derive any analytical option prices for the spark spread forwards. An alternative to geometric are additive models. One could model electricity and gas marginally by two correlated Ornstein-Uhlenbeck processes. In order to ensure positive prices, one has to use subordinators for the driving noise. Such models have been suggested and analysed in Benth, Kallsen and Meyer-Brandis (2005) for electricity, however, they turn out to be complicated to fit empirically. The pricing presented in this paper is easily modified to cover such model. Note that both functions μ and λ defined in Lemma 3.2 and Prop. 3.4, respectively, are functions involving the market price of risk μ_{mpr} , and observe that by definition the spread forward F_{spread} is a martingale with respect to Q .

In Benth and Koekebakker (2005) it is shown that there exists no analytical expression for the forward dynamics with delivery over a period, when the underlying spot price process follows a Schwartz model. The forward is not even a Markov process, but depends on all other forwards with shorter delivery periods in a quite complicated way.

Now, let $\tau_2 = \tau_1 + \Delta\tau$, where $\Delta\tau > 0$ is a fixed delivery period. Since $\delta(t, \tau_1, \tau_2) \rightarrow 0$ when $\tau_1 \rightarrow \infty$, it follows

$$\lim_{\tau_1 \rightarrow \infty} \left(F_{\text{spread}}(t; \tau_1, \tau_2) - \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du \right) = \frac{\mu_{\text{mpr}} - i\psi'(0)}{\alpha} .$$

Hence, the forward prices will converge to the average seasonal price modified by the market price of risk and the drift introduced by the Lévy jump process for deliveries starting in the far future. Thus, for delivery periods in the long end of the forward curve we approximately have

$$F_{\text{spread}}(t; \tau_1, \tau_2) \approx \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du + \frac{\mu_{\text{mpr}} - i\psi'(0)}{\alpha}.$$

We also note that the forward price becomes independent on the current spark spread and will become non-stochastic in the limit. This provides us with a simple tool to estimate the market price of risk from forward price data as long as the market trades in forwards where the delivery period is far into the future. We simply subtract the seasonal average from the observed forward price, and derive the market price of risk.

4 Pricing of spark spread options using Fourier analysis

In this section we analyse the option pricing dynamics written on the spark spread using Fourier analysis. For this purpose, consider an option written on the spark spread with payoff $f(F_{\text{spread}}(T, \tau_1, \tau_2))$ at the exercise time $T \leq \tau_1$ for some real-valued Borel measurable function f such that $f(F_{\text{spread}}(T, \tau_1, \tau_2)) \in L^1(Q)$. Denote the price at time $t \leq T$ of this option by

$$C_{\text{spread}}(t, T, \tau_1, \tau_2) = e^{-r(T-t)} \mathbb{E}_Q [f(F_{\text{spread}}(T, \tau_1, \tau_2)) | \mathcal{F}_t].$$

Since we shall be mostly interested in call options with payoff function $f(x) = \max(x - K, 0)$, we note that for such options the payoff function is at most linearly growing. The following Lemma tells us that options with at most linearly growing payoff functions are indeed Q -integrable under the hypothesis of square-integrability of the Lévy process L :

Lemma 4.1. *Suppose that $|f(x)| \leq c(1 + |x|)$ for some positive constant c . Then we have that $f(F_{\text{spread}}(T, \tau_1, \tau_2)) \in L^1(Q)$.*

Proof. If $F_{\text{spread}}(T, \tau_1, \tau_2) \in L^1(Q)$, then by the linear growth of f we have that the Lemma holds. We prove this.

From the dynamics in Prop. 3.4, we see that $F_{\text{spread}}(T, \tau_1, \tau_2) \in L^1(Q)$ whenever

$$\int_0^T e^{\alpha u} dL(u) \in L^1(Q).$$

Recall that the characteristics of L is the same under P and Q , and appealing to Cauchy-Schwarz' inequality and Itô's Formula for semimartingales, we find that the Lévy integral is integrable under Q whenever L is square integrable, which we have assumed in (3.5). \square

Let us first restrict our attention to payoff functions being integrable, i.e., assuming that $f \in L^1(\mathbb{R})$. Note that neither call nor put options are included in this case, but we can consider different types of knock-out options. However, call and put options can be analysed after an appropriate exponential damping, which we will consider in a moment.

Let \widehat{f} denote the Fourier transform of f , and \star be the convolution product. Recall from the inverse Fourier transform that

$$f(x) = \frac{1}{2\pi} \int \widehat{f}(y) e^{-ixy} dy.$$

We derive the following price dynamics for a spark spread option expressed in terms of an inverse Fourier transform:

Proposition 4.2. *Let $f \in L^1(\mathbb{R})$ and $0 \leq t \leq T \leq \tau_1$. If $f(F_{\text{spread}}(T, \tau_1, \tau_2)) \in L^1(Q)$, then the price dynamics*

$$C_{\text{spread}}(t, T, \tau_1, \tau_2) = C_{\text{spread}}(F_{\text{spread}}(t, \tau_1, \tau_2); t, T, \tau_1, \tau_2)$$

is given by

$$C_{\text{spread}}(x; t, T, \tau_1, \tau_2) = e^{-r(T-t)} \frac{1}{2\pi} \int \widehat{f}(y) \widehat{p}_{t,T,\tau_1,\tau_2}(y) e^{\Psi(t,T;-y\delta(T,\tau_1,\tau_2))} e^{-iyx} dy. \quad (4.1)$$

Here, p_{t,T,τ_1,τ_2} is the probability density of a normal random variable with expectation

$$\frac{-i\psi'(0)}{\alpha} \delta(0, \tau_1, \tau_2) (e^{\alpha T} - e^{\alpha t}).$$

and variance

$$\frac{\sigma^2}{2\alpha} \delta^2(T, \tau_1, \tau_2) (1 - e^{-2\alpha(T-t)}).$$

Furthermore, the function Ψ is defined in Lemma 3.2 and δ in Prop. 3.4.

Proof. From Prop. 3.4 we find,

$$\begin{aligned} & \mathbb{E}_Q [f(F_{\text{spread}}(T, \tau_1, \tau_2)) | \mathcal{F}_t] \\ &= \frac{1}{2\pi} \int \widehat{f}(y) \mathbb{E}_Q \left[e^{-iy \left(\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du + \lambda(T, \tau_1, \tau_2) + \delta(T, \tau_1, \tau_2) \widetilde{S}(T) \right)} | \mathcal{F}_t \right] dy \end{aligned}$$

$$= \frac{1}{2\pi} \int \widehat{f}(y) \mathbb{E}_Q \left[e^{-iy\delta(T, \tau_1, \tau_2) \widetilde{S}(T)} \mid \mathcal{F}_t \right] e^{-iy(\lambda(T, \tau_1, \tau_2) + \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du)} dy.$$

The Proposition follows from the key formula in Lemma 3.2. \square

The pricing dynamics of the spread option becomes independent of the market price of risk μ_{mpr} . It comes from a Girsanov change in the Wiener part of the spark spread dynamics only, a risk that can be hedged away. The jump risk, on the other hand, is not priced in our model and therefore there should be no dependency on any market price of risk in the option price dynamics. We also see that in the case of no jumps in the spark spread, i.e. $L = 0$, the expectation of the normal random variable becomes zero, in addition to $\Psi = 0$. The price can then be easily read off as an expectation of a function of a normal random variable with expectation equal to the forward spread price at time t and variance as in the Proposition.

We proceed to analyse call and put options from the Fourier perspective. Since the payoff function $f(x) = \max(x - K, 0)$ for a call option is not in $L^1(\mathbb{R})$, it is not covered by the above Proposition. However, following the idea in Carr and Madan (1999), we can treat the call option by similar tools as in Prop. 3.4 after an appropriate damping of the linear growth of the payoff. Introduce the following exponentially dampened payoff function

$$f_\gamma(x) = e^{-\gamma x} \max(x - K, 0) \quad (4.2)$$

for $\gamma > 0$, and note that

$$f(x) = \frac{e^{\gamma x}}{2\pi} \int \widehat{f}_\gamma(y) e^{-ixy} dy = \frac{1}{2\pi} \int \widehat{f}_\gamma(y) e^{(\gamma - iy)x} dy.$$

The function f_γ is obviously integrable for every positive constant γ , and the price dynamics of a call option is given by the following Proposition:

Proposition 4.3. *Suppose that the Lévy measure $\ell(dz)$ of L satisfies the exponential integrability condition*

$$\int_1^\infty e^{\gamma z} \ell(dz) < \infty$$

for some $\gamma > 0$. Then, the price of a call option with strike K at exercise time T can be represented as

$$C_{\text{spread call}}(x; t, T, \tau_1, \tau_2) = e^{-r(T-t)} c(t, T, \tau_1, \tau_2) \times$$

$$\frac{1}{2\pi} \int \widehat{f}_\gamma(y) \widetilde{p}_{t,T,\tau_1,\tau_2}^\gamma(y) \exp(\Psi(t, T; (-y - i\gamma)\delta(T, \tau_1, \tau_2))) e^{(-iy+\gamma)x} dy, \tag{4.3}$$

where f_γ is defined in (4.2),

$$c(t, T, \tau_1, \tau_2) = \exp\left(\frac{i\psi'(0)}{\alpha} \delta(0, \tau_1, \tau_2)(e^{\alpha T} - e^{\alpha t}) - \frac{\gamma^2 \sigma^2}{4\alpha}(1 - e^{-2\alpha(T-t)})\right)$$

and $p_{t,T,\tau_1,\tau_2}^\gamma$ is the probability density function of a normal random variable with expectation

$$\frac{-i\psi'(0)}{\alpha} \delta(0, \tau_1, \tau_2)(e^{\alpha T} - e^{\alpha t}) - \frac{\gamma \sigma^2}{4\alpha}(1 - e^{-2\alpha(T-t)})$$

and variance

$$\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(T-t)}).$$

The other functions are as in Prop. 4.2 above.

Proof. Denote by $f(x) = \max(x - K, 0)$. We have from the considerations above that

$$\begin{aligned} & \mathbb{E}_Q [f(F_{\text{spread}}(T, \tau_1, \tau_2)) | \mathcal{F}_t] \\ &= \frac{1}{2\pi} \int \widehat{f}_\gamma(y) \mathbb{E}_Q \left[e^{(-iy+\gamma)\left(\frac{1}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du + \lambda(T, \tau_1, \tau_2) + \delta(T, \tau_1, \tau_2) \widetilde{S}(T)\right)} | \mathcal{F}_t \right] dy \\ &= \frac{1}{2\pi} \int \widehat{f}_\gamma(y) \mathbb{E}_Q \left[e^{(-iy+\gamma)\delta(T, \tau_1, \tau_2) \widetilde{S}(T)} | \mathcal{F}_t \right] \\ & \quad \times e^{(-iy+\gamma)\left(\lambda(T, \tau_1, \tau_2) + \frac{1}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2} \Lambda(u) du\right)} dy. \end{aligned}$$

In order for the expectation to be well-defined, the random variable

$$\exp(\gamma \delta(T, \tau_1, \tau_2) \widetilde{S}(t))$$

must be integrable. However, it holds that

$$\mathbb{E}_Q \left[\exp\left(\gamma \int_0^t e^{-\alpha(t-u)} dL(u)\right) \right] = \exp\left(\int_0^t \psi(-i\gamma e^{-\alpha(t-u)}) du\right)$$

as long as $\psi(-i\gamma e^{-\alpha(t-u)})$ is integrable over $[0, t]$, which by definition of ψ and the exponential integrability assumption in the Proposition, holds true.

By extending the key formula in Lemma 3.2 to complex constants θ , and using $\theta = -iy + \gamma$, we derive the expression in the Proposition. \square

To price put options on the spread, we apply the relation

$$\max(x - K, 0) - \max(K - x, 0) = x - K,$$

to derive that

$$C_{\text{spread put}}(x; t, T, \tau_1, \tau_2) = C_{\text{spread call}}(x; t, T, \tau_1, \tau_2) + Ke^{-r(T-t)} - x, \quad (4.4)$$

with $x = F_{\text{spread}}(t, \tau_1, \tau_2)$. This is the put-call parity, which must hold in order to prevent arbitrage opportunities.

The Fourier transforms in Props. 4.2 and 4.3 are tailor-made for applying the fast Fourier transform (FFT) numerical technique. For given models of the jump processes (see the last Section for more details) we will frequently have an analytic expression of the integrand ψ of Ψ available. Thus, after a numerical or analytical integration the inverse Fourier transform can be calculated by FFT directly, as long as we know the Fourier transform of the payoff function f . In the next Lemma we state the Fourier transform of the payoff from a call option, which is derived by a straightforward integration:

Lemma 4.4. *The Fourier transform of the function f_γ for $\gamma > 0$ defined in (4.2) is given as*

$$\widehat{f}_\gamma(y) = \frac{e^{(iy-\gamma)K}}{\gamma^2 - 2iy\gamma - y^2}. \quad (4.5)$$

We refer to Carr and Madan (1999) for further discussions and numerical examples on the application of the FFT for valuation of option prices.

If we would have modelled gas and electricity spot prices separately as two exponential Ornstein-Uhlenbeck processes, we would have faced two problems. Firstly, the dynamics of the two forward price processes underlying the spark spread option would not have been analytically available, which means that we will have to evaluate numerically the forward price. Secondly, even if we would have the forward prices with delivery over a period available, we face the Fourier transform of a two dimensional payoff function, which is hard to derive for a call option with general strike price K . Thus, our modeling approach has a big advantage over marginal modeling of the two commodities underlying the spark spread option.

Let us briefly mention how we can represent the pricing formula in terms of convolutions products involving the option payoff function, the normal distribution and a jump distribution. For simplicity, we restrict our attention to integrable pay-off functions $f(x) \in L^1(\mathbb{R})$.

Using the definitions of ψ and Ψ , we have that

$$\begin{aligned} \exp(\Psi(t, T; -y\delta(t, \tau_1, \tau_2))) &= \exp\left(\int_t^T \psi(-y\delta(t, \tau_1, \tau_2)e^{-\alpha(T-u)})\right) \\ &= \mathbb{E}\left[\exp\left(-iy\delta(t, \tau_1, \tau_2) \int_t^T e^{-\alpha(T-u)} dL(u)\right)\right]. \end{aligned}$$

Hence, if we denote the probability distribution function of

$$-\delta(t, \tau_1, \tau_2) \int_t^T e^{-\alpha(T-u)} dL(u),$$

by $F_L(x)$, then the above calculation reveals that the Fourier transform of this measure is equal to

$$\exp(\Psi(t, T; -y\delta(t, \tau_1, \tau_2))).$$

Using this information, we can write the spread option price by the following convolution formula:

$$C_{\text{spread}}(x; t, T, \tau_1, \tau_2) = \int_{\mathbb{R}} \{f \star p_{t,T,\tau_1,\tau_2}\}(y-x) F_L(dy) \quad (4.6)$$

with p_{t,T,τ_1,τ_2} defined in Prop. 4.2. Thus, the price dynamics of a spark spread option can be interpreted as a weighted average of spark spread options written on a forward dynamics without jumps. The weighting is over a modification of the distribution of the jumps.

5 Modeling the spark spread for UK electricity and gas

We test our model on spot price data for gas and electricity observed in UK, and discuss different issues concerning the modeling of price spikes by jump processes.

5.1 Empirical analysis and model fitting

We had available electricity and gas spot prices recorded in the period 06 02 2001 to 27 04 2004 in United Kingdom. Observations are daily market prices, quoted at each working day. This resulted in 806 data over the period of observation. Electricity spot prices are recorded in £/MWh, while gas is in

pence/therm. Data are transformed to spark spread prices according to (2.1) by taking the difference of the electricity spot prices and the gas spot prices multiplied by heat rate c . The heat rate converts the gas prices into the units of electricity, and was chosen to be $c = 0.85$, which seems to be a reasonable number from practice. Spark spread differences are presented in Fig. 1. From this time series we see that in the first two years the spark spreads are less volatile than in the more recent part. This may be a sign that the market conditions changed, however, we shall not perform a more detailed analysis of this.

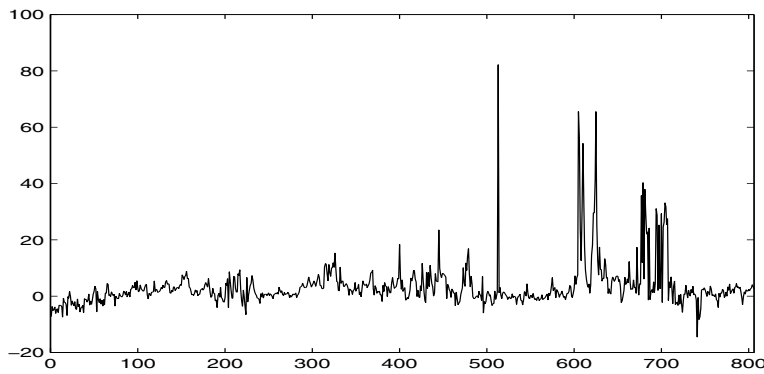


Figure 1: Spark spread differences.

To understand better the statistical properties of the spark spread differences, we build a histogram of empirical values and present it together with a fitted normal distribution density function in Fig. 2. The histogram shows obvious non-normality of data: it has an extremely high peak (kurtosis is equal to 30.22) and very heavy tails. Moreover, it has clear right skewness (skewness is equal to 4.55). Thus, it is not surprising that the normality hypothesis is rejected at the 1% level of significance.

We assume the seasonality function $\Lambda(t)$ to have the form $\Lambda(t) = \Lambda_1(t) + \Lambda_2(t)$, where

$$\Lambda_1(t) = a_0 + a_1 t \quad (5.1)$$

is the linear trend component, and

$$\Lambda_2(t) = b_0 + b_1 \cos(2\pi(t - b_2)/252) \quad (5.2)$$

describes seasonal effects. By simple linear regression, the slope and intercept were estimated to be $a_0 = 0.008$ and $a_1 = 0.273$, respectively. The value of

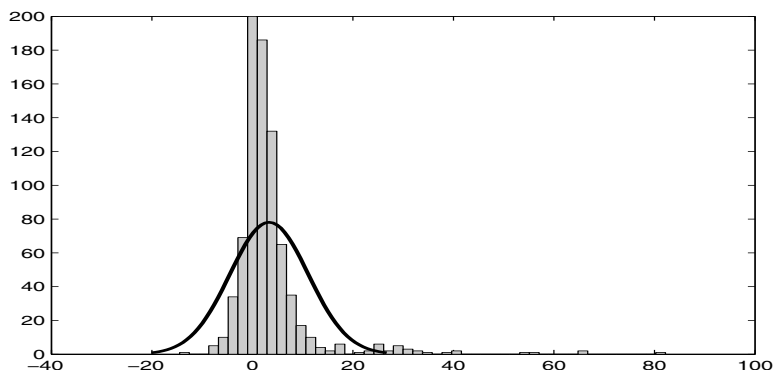


Figure 2: Histogram of the spark spread differences with normal density curve.

Table 1: Fitted parameters of seasonal function

| b_0 | b_1 | b_2 |
|-------|--------|--------|
| 0.180 | -3.307 | 11.526 |

intercept is statistically insignificant, however the slope is significant at the 1% level. Hence, during the period of observation there has been significant increase in values of the spark spread differences. The seasonality part $\Lambda_2(t)$ of $\Lambda(t)$ is fitted to the de-trended data using nonlinear least squares, implemented in Matlab as the function `nlinfit`. The estimated values are presented in Table 1. From Fig. 3 it is clear that there are quite weak seasonal effects in data. We checked for weekly, monthly and quarterly effects in spark spread differences as well, however they were basically not present and thus ignored. In the further analysis we remove the seasonality function $\Lambda(t)$ from the spark spread data, and consider the remaining time series.

To detect the mean-reversion in model (3.2), we regress today's data on the data of the previous day. The estimated mean-reverting parameter is $\alpha = 0.508$, being significant at the 1% level. Not unexpectedly, the fitted intercept appeared to be statistically insignificant, and is therefore assumed to be equal to zero as in the model (3.2). The R^2 after fitting the seasonal component and mean-reversion coefficient to data is 36%. This low value may be attributed to the heavy tailed residuals. The residuals from the regression analysis are presented as a time series in Fig. 4.

The histogram of the residuals of the spark spread differences with the fitted normal density curve is presented in Fig. 5, where we see that the residuals

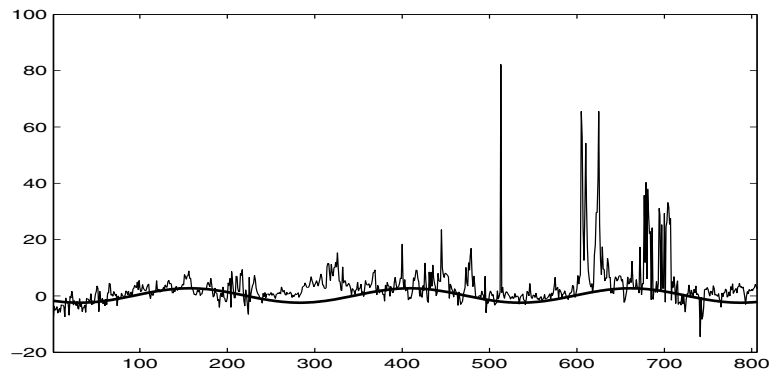


Figure 3: De-trended spark spread differences with the seasonality function.

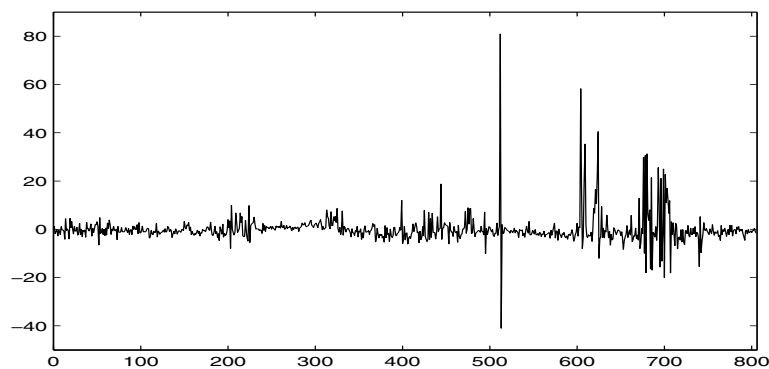


Figure 4: Spark spread differences residuals.

are far from being normally distributed. The residuals have mean close to zero, and standard deviation equal to 6.23. They have an extremely high peak (kurtosis is equal to 53.14) and very heavy tails. The skewness is equal to 4.80, indicating quite distinct right-hand skewness in the residuals. We continue our analysis by investigating different specifications of the Lévy process $L(t)$ in the spark spread dynamics (3.2) to fit these residuals.

An analytically tractable and flexible class of models for residuals having skewness and heavy tails is the normal inverse Gaussian (NIG) Lévy process. The NIG Lévy process $L(t)$ is defined by letting $L(1)$ be NIG distributed (and in addition putting $\sigma = 0$). This class of Lévy processes was applied to oil and gas data in Benth and Šaltytė-Benth (2004), and is a candidate for

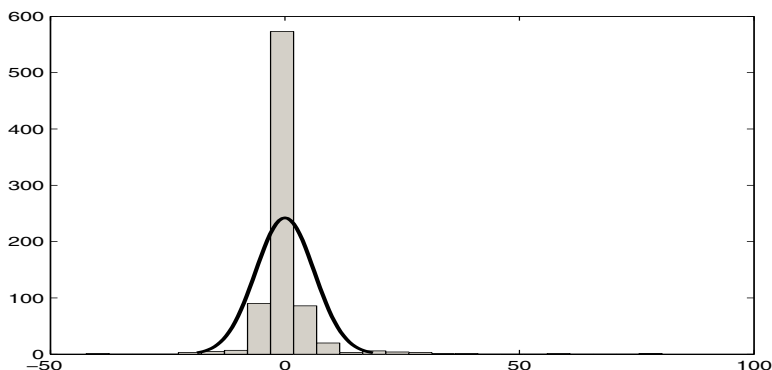


Figure 5: The histogram of the spark spread differences residuals.

modeling the spark spread residuals as well (see Barndorff-Nielsen (1998) and Barndorff-Nielsen and Shephard (2001) for applications to finance). However, we shall in this paper follow a different, but more traditional way of modeling jumps in a commodity time series. Observing that the residuals are mostly having small fluctuations around zero, but from time to time showing rather extreme jumps, it may be natural to consider the Lévy process $L(t)$ to be a pure jump process. A model of this form is conveniently fitted using the method of recursive filtering (see e.g. Clewlow and Strickland (2000) for a description of this algorithm), which we now apply to our data set.

We begin by calculating the empirical standard deviation of the residuals, being equal to 6.23. Then we identify all residuals larger in absolute value than $3 \times 6.23 = 18.68$ as jumps. In the first step, we find that there are 16 positive and 2 negative values that exceed this limit. We calculate the daily jump frequency by dividing the number of jumps by the total number of data, which results in the value 0.0223. After removing these jumps, we recalculate the standard deviation (becoming 3.36), and repeat the procedure of identifying jumps. We iterate this algorithm until the standard deviation of the residuals with jumps removed converges and no new jumps are identified. The results of the recursive filtering procedure are presented in Table 2.

Let us first discuss the “normal variations” given by the filtered residuals. From Table 2 we can read off the standard deviation (or volatility) of the filtered residuals to be $\sigma = 2.07$, and a histogram of normalized residuals is given in Fig. 6. It has a clear bell shape, with kurtosis and skewness being 0.32 and 0.22, respectively. The Kolmogorov-Smirnov statistics is significant at the level of 8%, meaning that the distribution is very close to normal. These facts

Table 2: Summary of the recursive filtering procedure

| Iteration | Std.dev. | Cumul. # jumps | Daily jump frequency |
|-----------|----------|----------------|----------------------|
| 1 | 3.36 | 18 | 0.0223 |
| 2 | 2.68 | 34 | 0.0422 |
| 3 | 2.35 | 50 | 0.0620 |
| 4 | 2.22 | 58 | 0.0720 |
| 5 | 2.13 | 64 | 0.0794 |
| 6 | 2.08 | 68 | 0.0844 |
| 7 | 2.07 | 69 | 0.0856 |
| 8 | 2.07 | 69 | 0.0856 |

indicate that the choice of Brownian motion as an error process is reasonable.

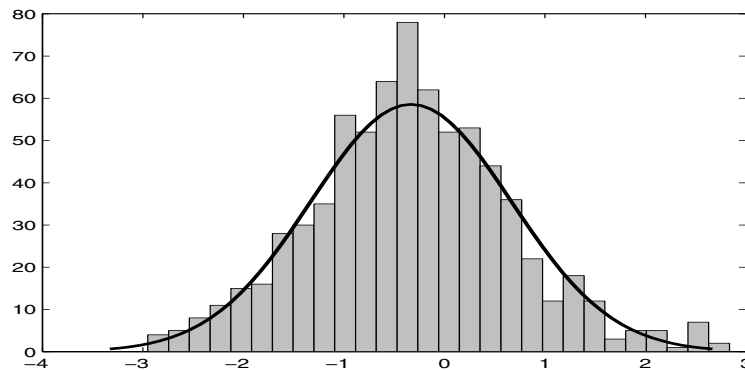


Figure 6: Histograms of the filtered residuals.

We next turn our attention to the filtered jumps. A suitable model for $L(t)$ is a compound Poisson process defined as

$$L(t) = \sum_{i=1}^{P(t)} X_i \quad (5.3)$$

for an independent sequence of *i.i.d* random variables X_i and $P(t)$ being a Poisson process with jump intensity λ . The jump intensity is read off from Table 2 as $\lambda = 0.0856$. We did not detect any seasonal nor weekly pattern of the jump intensity in our data set, which supports the choice of a constant

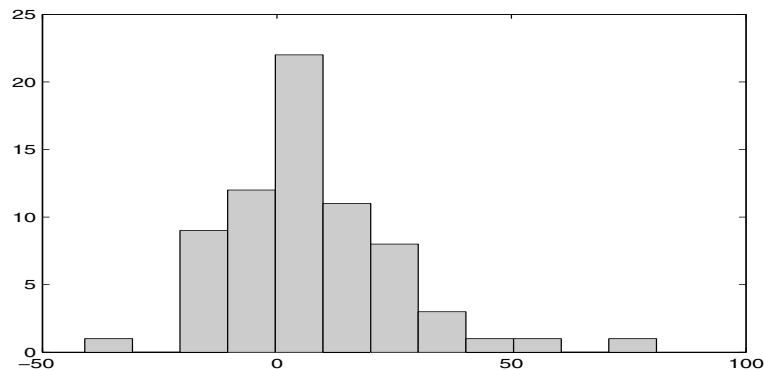


Figure 7: Histograms of the values of jumps

λ . In Fig. 7 we have plotted the histogram of the filtered jumps. To have an analytically tractable model (see Subsection 5.2 below for a discussion), we assume that the jumps are normally distributed, with estimated mean equal to 7.33 and standard deviation 18.78. Such an assumption seems to be reasonable, since Kolmogorov-Smirnov statistics is significant at the level of 6%. Hence, we model X_i as a normal random variable with the specified mean and standard deviation. We remark that the choice of $L(t)$ in (5.3) gives an additive version of the jump dynamics model for asset prices suggested by Merton (1976).

One may not be satisfied with modeling the jumps by a normally distributed random variable. Indeed, from Fig. 8, we see that the histograms for the negative and positive jumps are different. Moreover, the jump intensities are estimated to be 0.0273 for the negative jumps, while the positive have an intensity of 0.0583, which means that they occur more than twice as often. Such a skewness in jump behaviour occurs for other data series as well (see e.g. Clewlow and Strickland (2000) for more on this), and a natural model may be

$$L(t) = N^+(t) - N^-(t) \quad (5.4)$$

where $N^\pm(t)$ are two independent compounded Poisson processes, defined as

$$N^\pm(t) = \sum_{i=1}^{P^\pm(t)} X_i^\pm$$

for two independent sequences of *i.i.d* lognormal variables X_i^\pm with parameters s^\pm (shape) and m^\pm (scale). The Poisson processes $P^\pm(t)$ are assumed

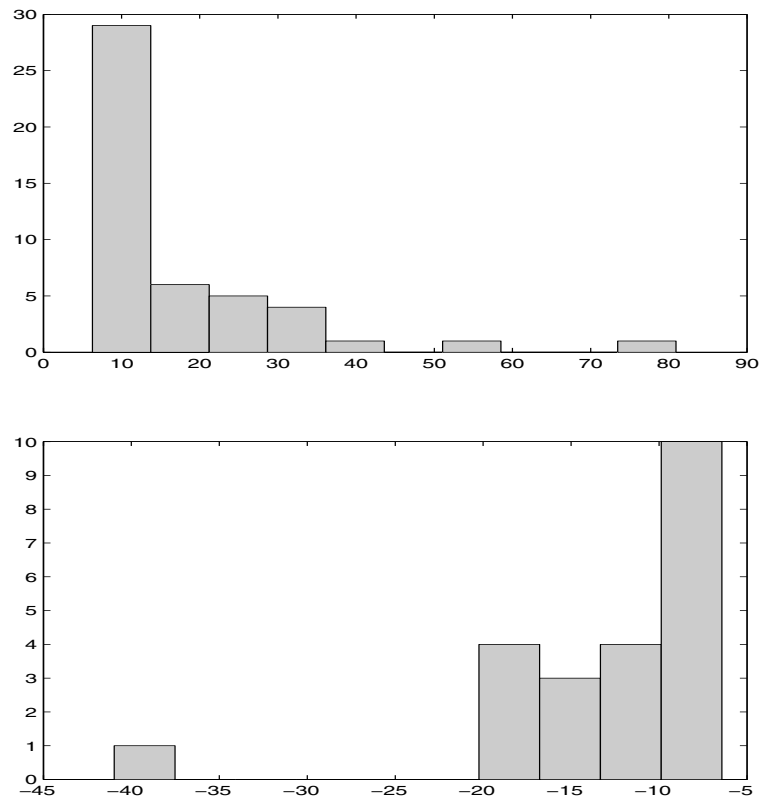


Figure 8: Histograms of the values of positive (top) and negative (bottom) jumps.

to have jump intensities λ^\pm , which we have estimated to be $\lambda^+ = 0.0583$ and $\lambda^- = 0.0273$. The estimated shape and scale parameters of the positive and negative jump size distributions are presented in Table 3. In both cases the Kolmogorov-Smirnov statistics is insignificant at 5% level. The unfortunate effect of choosing lognormal distributions for the jump sizes is that the integrability condition for call options does not hold.

We have followed the Box-Jenkins approach to fit our Ornstein-Uhlenbeck model to spread data. An alternative to this is to fit the mean-reversion parameter α in (3.2) by appealing to the autocorrelation structure of $S(t)$ in (3.1). It holds that the stationary covariance function of $S(t)$ is

$$\text{Cov}(S(t + \tau), S(t)) = e^{-\alpha\tau} \text{Var}(S(t))$$

and thus we can obtain α by a calibration to the empirical covariance and

Table 3: Estimated parameters of jumps distribution

| Parameter | Positive jumps | Negative jumps |
|-----------|----------------|----------------|
| m | 13.25 | 21.27 |
| s | 0.64 | 1.52 |

variances of the spread data. The next step is to fit the linear and seasonal trends using least squares. With the deterministic seasonality function and the mean-reversion available, we find that $\tilde{s}(t+1) - e^{-\alpha}\tilde{s}(t)$ are independent and identically distributed, and maximum-likelihood or Fourier inversion techniques (see Singleton (2001)) may be used for estimation. This approach was followed by Lucia and Schwartz (2002) in fitting an exponential Ornstein-Uhlenbeck process to electricity data.

5.2 Discussion of option pricing

Let us discuss the relation to option pricing of the proposed jump processes. Consider first the jump model stated in (5.3). If the cumulant function of X , the jumps size, is denoted by ϕ , we have that

$$\psi(\theta) = \ln \mathbb{E}[\exp(i\theta L(1))] = \lambda (e^{\phi(\theta)} - 1) .$$

When X is supposed to be a normally distributed random variable with expectation m_j and variance σ_j^2 , its cumulant becomes

$$\phi(\theta) = i\theta m_j - \frac{1}{2}\theta^2 \sigma_j^2 .$$

Noting that the exponential integrability condition in Prop. 4.3 is equivalent with $L(1)$ having exponential moments, that is, the expectation of $\exp(\gamma L(1))$ being finite, we see that for the above jump model we can price call options on the spark spread using the Fourier approach.

Turning our attention to the model (5.4), we find that

$$\psi(\theta) = \lambda(e^{\phi^+(\theta)} + e^{\phi^-(\theta)} - 2) ,$$

where $\phi^\pm(\theta)$ are the cumulant functions for X^\pm , respectively. We can calculate these for the lognormal model, ending up with an infinite series representation. However, we observe that exponential moments do not exist for the Lévy process L when X is lognormally distributed, and thus we can not use this model to price call options by FFT.

In order to apply the fast Fourier technique, we need the function Ψ , which is given as an integral of the cumulant ψ (see Lemma 3.3). In practical applications of the models proposed above, we need to approximate Ψ by a numerical integration procedure since the integral can not be analytically calculated.

At the time of writing, there exist only OTC trading of spark spread options¹. Since there is a lack of liquid prices taken from an organized market, it is of importance to have available tools yielding fast and reliable prices. We believe that the above framework may serve as such a tool. From the liquid forward markets of gas/coal and electricity, we can read off the market prices of risk. Further, our underlying stochastic model on the spark spread is estimated to historical data, which together with the market price of risk give all the required input to price the spark spread options efficiently. Reliable routines for marking-to-market and Value-at-Risk evaluations may be implemented based on our pricing technique.

6 Conclusions

We propose to model the spark spread directly by an Ornstein-Uhlenbeck process with seasonal trend and jumps, and show that this model allows for analytical forward prices when the underlying commodities deliver over a period. Based on this model, we can derive expressions for the price dynamics of spark spread option based on the inverse Fourier transform. FFT techniques can be used for fast and efficient numerical pricing. An empirical study of UK electricity and gas spot prices suggests that it is reasonable to model the price difference by an Ornstein-Uhlenbeck process.

In the electricity and gas markets forward contracts deliver over periods rather than at fixed points in time. Thus, in order to price spark spread options, we need to have available a dynamics for the underlying forwards. Modeling the spot price of electricity and gas with the more traditional exponential models will only lead to integral representations of the forward dynamics, and one is forced to use numerical techniques like Monte Carlo for general option pricing. Our modeling perspective suggests a way around this problem.

In a further study one may look at the problem of calibrating our model to the difference of two exponential Ornstein-Uhlenbeck process, which are marginally fitted to electricity and gas. In this way one can link to parameters of our model to the marginal behaviour of the two underlying commodities.

¹NYMEX has a market for *crack spreads*, a relative of the spark spread based on the difference of crude oil and a refined product

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