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Variance Gamma Option Model [1]

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Abstract

This document provides a brief description of VG model and valuation of European, Bermudian and American options in the framework of this model.

1 Variance Gamma Model

The Variance Gamma (VG) process was proposed see (Madan, Carr and Chang 1998) [1] to describe stock price dynamics instead of the Brownian motion in the original Black-Scholes Model. Two new parameters: θ skewness and ν kurtosis are introduced in order to describe asymmetry and 'fat' tails of real life distributions. The VG process is defined by evaluating Brownian motion with drift at a random time specified by gamma process. Let

$$b(t;\theta,\sigma) = \theta t + \sigma W(t)$$

where $W(t)$ is a standard Brownian motion. The VG process $X(t;\sigma,\nu,\theta)$ is defined as

$$X(t;\sigma,\nu,\theta) = b(\gamma(t;1,\nu);\theta,\sigma)$$

where $\gamma(t;1,\nu)$ is the gamma process with unit mean rate and density function given by

$$f(g) = \frac{g^{\frac{t}{\nu}-1}\exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}.$$

The probability density function for the VG process may be written as

$$h_t(x) = \int_0^\infty \frac{\mathrm{d}g}{\sqrt{2\pi g}} \exp(-\frac{(x-\theta g)^2}{2\sigma^2 g}) \frac{g^{\frac{t}{\nu}-1}\exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}, \tag{1}$$

or after integration over g

$$h_t(x) = \frac{2\exp(\frac{\theta x}{\sigma^2})}{\nu^{\frac{t}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{t}{\nu})} \left(\frac{x^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}} K_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2(\frac{2\sigma^2}{\nu}+\theta^2)}\right), \tag{2}$$

where K is the modified Bessel function of the second kind. The characteristic function $\phi_t(u)$ for the VG process has remarkably simple form:

$$\phi_t(u) \equiv \langle e^{iux} \rangle \equiv \int_{-\infty}^{+\infty} h_t(x) e^{iux} \mathrm{d}x = \left(\frac{1}{1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2}\right)^{\frac{t}{\nu}}. \tag{3}$$

The risk neutral process for stock price dynamics is given by

$$S(t) = S(0)\exp(rt + X(t;\sigma,\nu,\theta) + \omega t), \quad \omega = \frac{1}{\nu}\ln(1 - \theta\nu - \frac{1}{2}\sigma^2\nu), \tag{4}$$

where r is risk free rate and ω is defined in a way that the following relation holds:

$$\langle S(t) \rangle = e^{rt} S(0), \tag{5}$$

i.e. $\langle e^X \rangle = e^{-\omega t}$. Risk neutral parameters θ, ν, σ in Eq.(4) do not have to be equal to their statistical counterparts.

1.1 Statistical Parameters of the VG Distribution

Statistical parameters of VG distribution may be calculated from the historical data of the stock prices. In particular we have to find the the values of the parameters $\theta^\star$, $\nu^\star$ and $\sigma^\star$ such that the following expression is maximized:

$$\prod_{j=1}^{n} h_{\tau_j}(x_j) \tag{6}$$

where $h_{\tau_j}(x_j)$ are given by Eq.(2) and x_j are observed returns per time τ_j

$$x_j = \log(\frac{S_j}{S_{j-1}}). \tag{7}$$

1.2 Black-Scholes Limit

VG distribution has three parameters skewness θ, kurtosis ν and volatility σ. Kurtosis measures deviation from the Gaussian tail, *i.e.* in the limit $\nu \to 0$ as can be seen from Eq.(3):

$$\phi_t(u) \to \exp(-\frac{1}{2}\sigma^2 u^2 t + i\theta u t), \tag{8}$$

and probability distribution function (PDF) is shifted Gaussian with asymmetry measured by θ.

2 Option Pricing

2.1 European Options

The value of European option on a stock when the risk neutral dynamics is given by Eq. (4) is

$$V = \exp(-rT) \int_{-\infty}^{\infty} h_T(x - (r - q + \omega)T) W(e^x) dx, \tag{9}$$

where T is time until expiration, q is continuous dividend and $W(e^x)$ is payoff function that has the following form[2]

[2]We have to restrict the parameters of VG distribution in order to keep all moments of distribution finite. The restrictions can be derived from asymptotic behavior of Bessel

$$W(e^x) = (S_0 e^x - K)^+ \qquad \text{CALL}$$

$$W(e^x) = (K - S_0 e^x)^+ \qquad \text{PUT.}$$

Define effective drift $\mu = r - q + \omega$. Note[3] that Eq. (9) may be rewritten in the following form

$$V = \int_{-\infty}^{\infty} \prod_{j=1}^{n} dx'_j \exp(-r(T_j - T_{j-1})) h_{T_j - T_{j-1}}(x'_j - \mu(T_j - T_{j-1})) W(e^{x'_1 + \ldots + x'_n}), \tag{10}$$

where $T_n \equiv T, T_0 \equiv 0$.Finally we note that direct calculation allows us to derive the put call parity relation identical to Black - Scholes case:

$$C = S_0 e^{-qT} - K e^{-rT} + P, \tag{11}$$

where C is call value and P is put value.

2.2 Bermudian and American Options

Its convenient to change variables in Eq. (10)

$$x_1 \to x'_1, x_2 \to x'_1 + x'_2, x_n \to x'_1 + \ldots + x'_n,$$

so that Eq. (12) looks like

$$V = \int_{-\infty}^{\infty} \prod_{j=1}^{n} dx_j \exp(-r(T_j - T_{j-1})) h_{T_j - T_{j-1}}(x_j - x_{j-1} - \mu(T_j - T_{j-1})) W(e^{x_n}), \quad x_0 \equiv 0 \tag{12}$$

Bermudian options are valued using backward time iteration. Consider for example situation where only one exercise date T_1 is allowed between $T_0 = 0$ and $T_2 = T$(expiration time) so that the value $V(S_0)$ of Bermudian option is defined as

$$V(S_0) = \exp(-rT_1) \int h_{T_1}(x_1 - \mu(T_1)) f(x_1) dx_1, \tag{13}$$

function which appears in Eq.(2):

$$K_\nu(z) \to \sqrt{\frac{\pi}{2z}} e^{-z}, \quad z \to +\infty$$

or from requirement that $\omega = \frac{1}{\nu} \ln(1 - \theta\nu - \frac{1}{2}\sigma^2\nu)$ is well defined *i.e.* $\theta\nu + \frac{1}{2}\sigma^2\nu < 1$

[3]Here we use the fact that probability density function of VG model is stable under convolution *i.e.* $\int dy h_{t_1}(x - y) h_{t_2}(y - z) = h_{t_1 + t_2}(x - z)$ In Fourier space this conditions takes very simple form $\phi_{t_1}(k)\phi_{t_2}(k) = \phi_{t_1 + t_2}(k)$ where $\phi_t(k)$ is characteristic function

where

$$f(x_1) = \max(W(e^{x_1}), \exp(-r(T_2 - T_1)) \int h_{T_2 - T_1}(x_2 - x_1 - \mu(T_2 - T_1)) W(e^{x_2}) dx_2). \tag{14}$$

The generalization for many dates is straitforward. American options are obtained from Bermudian using Richardson extrapolation.

2.3 FFT Method

FFT algorithm is effective way to compute the fourier transform. We compute convolutions in Eqs. (10, 13) using FFT in three steps:

1) compute fourier transform of $f(x_l)$

2) multiply characteristic function and fourier image of $f(x_l)$

3) apply inverse fourier transform

2.4 Barrier Options

Straitforward application of FFT method to barrier options leads to the problem of very bad convergence[4]. In particular we have to evaluate here the integrals of the following form

$$\int_{-\infty}^{\infty} h(x-y) f(y) dy, \tag{15}$$

where function $f(y)$ may have discontinuities in one (single barrier) or too points (double barriers). The bad convergence is caused by these discontinuities. In order to deal with this problem we use the following trick. Let $y_\star$ be the point where the function $f(y)$ has discontinuity, *i.e.*

$$f(y) \equiv f_1(y), y < y_\star$$

$$f(y) \equiv f_2(y), y > y_\star.$$

Define the following functions

$$\tilde{f}_1(y) \equiv f_1(y), y < y_\star$$

$$\tilde{f}_1(y) \equiv f_1(y_\star), y > y_\star$$

$$\tilde{f}_2(y) \equiv f_2(y_\star), y < y_\star$$

$$\tilde{f}_2(y) \equiv f_2(y), y > y_\star$$

[4]The required number of points in order to get correct results was around $N = 2^{16} = 65536$

note that these functions are continuous at $y_\star$ by construction. Then one may write

$$\int_{-\infty}^{\infty} h(x-y)f(y)dy = \int_{-\infty}^{\infty} h(x-y)(\tilde{f}_1(y)+\tilde{f}_2(y))dy - (f_1(y_\star)-f_2(y_\star))H(x-y_\star), \tag{16}$$

where $H(x)$ is cumulative distribution function

$$H(x) \equiv \int_{\infty}^{x} h(z)dz.$$

Now we isolated all the "singularities" in the second term of Eq.(16) and the first term may be calculated with much smaller number of points. This procedure may be easily generalized to the case when we have discontinuity more than in one point.

References

[1] D. B. Madan, P. P. Carr & E. C. Chang 1998 "The variance gamma process and option pricing." European Finance Review, v. 2, p. 79-105.

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