

# Bloomberg

## FX Market Behavior and Valuation

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# Outline

- 1 Introduction: From interest rates to exchange rates.
- 2 Vanilla currency options
- 3 Barrier Options

## Introduction: From interest rates to exchange rates.

Markets consist of sets of traded securities, each with a price process. To work with FX options, we need to identify the appropriate securities and options.

A Two Currency Model:

- $Z_d(t, T)$  — A system of zero coupon bonds in the domestic currency.
- $Z_f(t, T)$  — A system of zero coupon bonds in the foreign currency.

What's missing?

- Self consistent, but
- no trading can occur between traders of domestic bonds and traders of foreign bonds.

Not quite a market if one can't sell one security to buy the other.

# The missing piece — the exchange rate

To complete the market definition, we need:

- $X_{d/f}(t)$  — The time  $t$  cost in the domestic currency of one unit of foreign currency.

Now the value of a portfolio with positions  $a_{d,T_i}(t)$  in domestic bonds with maturities  $T_i$  and positions  $a_{f,T'_i}(t)$  in foreign bonds with maturities  $T'_i$  has value (expressed in the domestic currency):

- $\sum a_{d,T_i}(t)Z_d(t, T_i) + X_{d/f}(t) \sum a_{f,T'_i}(t)Z_f(t, T'_i)$ .

Or, similarly, its price in the foreign currency is:

- $\sum a_{d,T_i}(t)Z_d(t, T_i)/X_{d/f}(t) + \sum a_{f,T'_i}(t)Z_f(t, T'_i)$ .

However, note:

- Securities in this model are still bonds:
  - Domestic bonds with price process  $Z_d(t, T)$ .
  - Foreign “bonds” with price process  $Z_f(t, T)X_{d/f}(t)$ .
- These are the securities that can be bought, sold, and owned.
- There's no security with price process  $X(t)$ .

# Vanilla currency options

Adding the exchange rate adds more than just being able to exchange foreign and domestic bonds.

- Foreign exchange spot market —  $X_{d/f}$ .
- Foreign exchange forwards market — forwards on  $X_{d/f}$ .
- Foreign exchange options — options on  $X_{d/f}$ .

But, given that  $X$  isn't a security in our market, how can we have forwards and options on  $X$ ?

- How can we realize forward contracts if  $X$  can't be traded?
- Options are contingent claims on securities. If  $X$  isn't a security, what's a call on it?

## Forwards in the FX market

Even without trading  $X$ , we can still work out forward prices.

The time  $t$  forward price for maturity  $T$  is the exchange rate  $K$  at which we'd agree at time  $t$  to convert funds at  $T$ .

At time  $t$  we can lock in an exchange rate at time  $T$  by buying and selling zero coupon bonds.

- Buy one foreign bond maturing at  $T$ , at a cost of  $Z_f(t, T)X(t)$ .
- Sell domestic bonds to cover the cost. Need  $Z_f(t, T)X(t)/Z_d(t, T)$  domestic bonds.
- At  $T$ , we get 1 unit of foreign currency from our foreign bond, and pay  $Z_f(t, T)X(t)/Z_d(t, T)$ .

At time  $t$ , we've transacted at zero cost. Our only P&L occurs at time  $T$ , when we pay  $Z_f(t, T)X(t)/Z_d(t, T)$  for maturation of the domestic bond we sold, and receive 1 unit of foreign currency for the foreign bond that we bought. This makes  $Z_f(t, T)X(t)/Z_d(t, T)$  the forward exchange rate at  $t$  for time  $T$ .

## Call options

What's a call option on  $X$  with strike  $K$  and maturity  $T$ ?

- At time  $T$  one has the option to buy one unit of foreign currency at the exchange rate  $K$ .
- If  $X_{d/f}(T) > K$ , then buy one unit for  $K$ , then convert it back on the open market for  $X_{d/f}(T)$ , making  $X_{d/f}(T) - K$  in the process.
- If  $X_{d/f}(T) \leq K$ , then we don't exercise the option.
- Payoff =  $(X_{d/f}(T) - K)^+$  in domestic currency.

But, again, how can we have a contingent claim on something that's not a traded security?

## Call options as contingent claims

Although we can't own shares in the exchange rate, it does show up as the value of a security. We can exploit this to express a call as a contingent claim.

- The foreign ZCB with maturity  $T$  costs  $Z_f(t, T)X(t)$  at time  $t$ .
- At time  $T$ , it costs  $Z_f(T, T)X(T) = X(T)$ .

So, the exchange rate is the domestic price of a foreign zero coupon bond at maturity, and a call option with maturity  $T$  is actually a call on the domestic price of the foreign ZCB maturing at  $T$ .

## Call option pricing — Black-Scholes case

To price options, consider the domestic and foreign forward rates ( $f_c(t, T)$ , for  $c = d$  or  $f$ ), short rates ( $r_c(t)$ ) and money market accounts ( $B_c(t)$ ):

$$\begin{aligned}f_c(t, T) &= -\partial \ln Z_c(t, T) / \partial T \\r_c(t) &= f(t, t) \\B_c(t) &= e^{\int_0^t r_c(t) dt} \\Z_c(t, T) &= \mathbf{E}_t^{Q_c}[1/B_c(T)] B_c(t) \\&= \mathbf{E}_t^{Q_c}[e^{-\int_t^T r_c(t) dt}]\end{aligned}$$

where  $Q_c$  is the equivalent Martingale measure for the money market numeraire (restricting attention just to the market for securities in currency  $c$ ).

## Value of an FX option

In this framework, the foreign and domestic money market accounts in the domestic currency are:

- $B_f(t)X(t)$
- $B_d(t)$

If  $B_d$  and  $B_f$  are deterministic, and  $X(t)$  is a lognormal process with constant drift and volatility, then this is exactly the setting for Black-Scholes on a dividend paying stock with proportional dividends paid at rate  $r_f(t)$ , so for constant rates, we get the value of the call option with strike  $K$  and maturity  $T$  being:

$$\begin{aligned}C(X(0), T) &= e^{-r_d T} (FN(d_1) - KN(d_2)) \\F &= X(0)e^{(r_d - r_f)T} \\d_1 &= (\ln(F/K) + (\sigma^2/2)T)/(\sigma\sqrt{T}) \\d_2 &= d_1 - \sigma\sqrt{T}\end{aligned}$$

We see that  $F$  is the forward price of  $X$  at maturity  $T$ , and actually, the forward based version of Black-Scholes is identical for dividend paying stocks and exchange rates.

## Effects of stochastic rates

More generally, if rates are stochastic, under the equivalent Martingale measure for numeraire  $B_d$ , we have that the call option value at time  $t$  is given by the discounted expected value

$$V(t) = \mathbf{E}_t^Q[B_d(t)/B_d(T)(X(T) - K)^+]$$

If  $B_d$  is independent of  $X$ , this becomes:

$$\begin{aligned} V(t) &= \mathbf{E}_t^Q[B_d(t)/B_d(T)]\mathbf{E}_t^Q[(X(T) - K)^+] \\ &= Z_d(t, T)\mathbf{E}_t^Q[(X(T) - K)^+] \end{aligned}$$

which gives us the same value for the option at time zero as in the model with deterministic domestic interest rates (ignoring the impact of the drift on  $Q$ ).

So, we see that in the case of a lognormal  $X(t)$ , if the domestic interest rates are constant or independent of the “dividend paying stock”, then the Black-Scholes formula still holds.

## Need for stochastic interest rates

When can we ignore stochastic interest rates?

- To the extent that interest rates are almost constant (low volatility)
- To the extent that interest rates are reasonably independent of exchange rates (short time horizons), the Black-Scholes formula will still be a good approximation of the model price.

Under these conditions, we can work directly with the volatility of the exchange rate.

How can we avoid worrying about stochastic interest rates?

- By throwing away the model — modeling the option prices (or the implied volatility surface) directly.

When would we need stochastic rates?

- Modeling interactions between rates and FX — hybrid products, etc.
- Estimation of long dated option prices.

## Breaking Black-Scholes

To some extent, this is all moot. Black-Scholes doesn't hold for rates, and doesn't hold for exchange rates either. Not only does implied volatility change with option maturity, it also varies with strike:

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### Currency Volatility Surface

Save Send Download Options 3D Graph \* Composite (CMPN) EURUSD

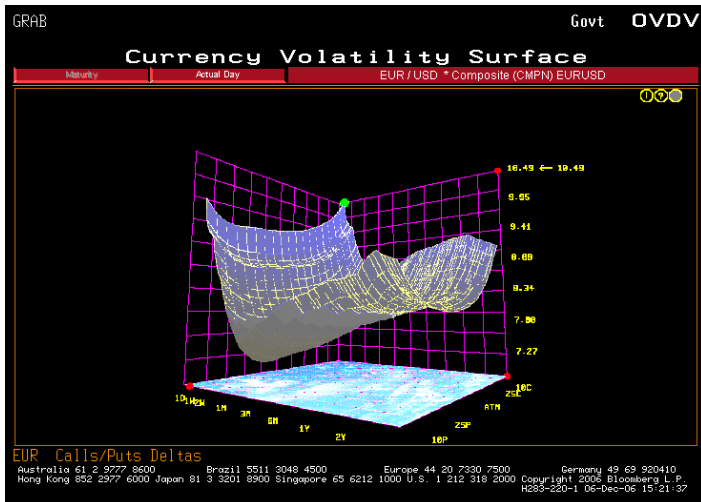
Currencies: **EUR-USD** Date: **12/ 6/06**  
 EUR Calls/Puts Deltas Format: **1** RR/BF  
 Calendar: **2** None Side: **1** Bid/Ask

EXP	ATM(50D)		25D RR		25D BF		10D RR		10D BF	
	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
1D	8.378	10.633	0.396	0.396	-0.162	0.384	0.713	0.713	-1.000	1.768
1W	8.378	9.226	0.396	0.396	0.010	0.212	0.713	0.713	-0.030	0.960
2W	7.851	8.352	0.396	0.396	0.051	0.172	0.713	0.713	0.192	0.737
3W	7.403	7.799	0.495	0.495	0.162	0.242	0.891	0.891	0.293	0.707
1M	7.313	7.588	0.594	0.594	0.131	0.172	1.069	1.069	0.414	0.414
2M	7.413	7.688	0.594	0.594	0.131	0.172	1.069	1.069	0.444	0.697
3M	7.512	7.789	0.594	0.594	0.152	0.202	1.069	1.069	0.444	0.707
4M	7.642	7.919	0.584	0.584	0.162	0.222	1.039	1.039	0.455	0.727
6M	7.711	7.990	0.574	0.574	0.182	0.222	1.020	1.020	0.495	0.727
9M	7.741	8.010	0.554	0.554	0.182	0.222	1.000	1.000	0.515	0.758
1Y	7.821	8.080	0.545	0.545	0.182	0.232	0.980	0.980	0.535	0.758
18M	7.811	8.191	0.545	0.545	0.182	0.232	0.980	0.980	0.465	0.798
2Y	7.851	8.251	0.545	0.545	0.172	0.242	0.980	0.980	0.434	0.808
5Y	8.109	8.109	0.347	0.347	0.162	0.232	0.624	0.624	0.404	0.818
10Y	7.500	8.500	0.347	0.347	0.162	0.232	0.624	0.624	0.404	0.818

\*Default RR = EUR Call - EUR Put Interpolated points  
 Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410  
 Hong Kong 852 2977 6000 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2006 Bloomberg L.P.  
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# Breaking Black-Scholes

Or, graphically:



## Dealing with reality

Given that even vanilla FX options are trading far from a constant vol, how do we price them?

- Find a model that matches the market quotes.
- Interpolate.

There are models which will match the market quotes (e.g. local vol with a sufficiently high dimensional parameterization of the vol surface), but in that case, one is essentially using the model to interpolate and extrapolate. For vanilla pricing, there's substantially more complexity and not much additional value.

The value of a model comes more for hedging and exotics pricing (mostly exotics pricing).

# Volatility Surface Interpolation

If Black-Scholes doesn't work, how do we price vanillas? By interpolation on BS vols:

- Interpolate.
- Handle business time.
- Take care of the wings.

# Interpolation

## Interpolation method

- Interpolate implied volatility in delta and maturity.
- Use cubic splines in delta.
- Use linear interpolation of realized variance in time.

Interpolating realized variance is the same as using a piecewise constant instantaneous volatility fitted to the quoted maturities.

### Complications:

- Quotes are often missing — we don't have a full table.

### Solution:

- Fit a constrained surface to the quotes.
- Use the fit to fill in the holes.
- Then interpolate.

After testing various functional forms on a variety of input data, we settled on using  $Ad_1 + Bd_1^2 + C\sqrt{T}$ . Filling the holes directly from the fitted form would cause jumps. To correct for this, we interpolate on the errors as well.

## Business time

Another market feature is that implied volatility accrues differently on different days. Knowing that volatility is lower on weekends and holidays, and higher prior to important scheduled announcements (such as election results), we want to take this into account in our interpolation.

- Assign a weight  $w_i \geq 0$  to each day.
- Distribute the change in realized variance for the period between bracketing quoted maturities according to the weights, giving the  $i^{\text{th}}$  day  $w_i / \sum w_j$  of the change.

Assigning a weight of zero to a day causes zero volatility for that day, and increases the realized variance for the remaining days of the period to make up for it.

## Winging it

We also need to price options at deltas outside the range of quoted deltas, and at maturities outside the range of quoted maturities.

- Outside the quoted range, in lieu of information on exactly how fast vols should change, we use flat extrapolation.

One could argue that in strike, one could estimate the wing shape by fitting an appropriate form to the observed points. But option prices are fairly insensitive to changes in implied vol at strikes beyond the extreme quoted strikes, so it doesn't matter so much.

## Extrapolating in maturity

In maturity, it's harder to say how volatilities should rise or fall, and volatility tends to be fairly flat towards the end of the quoted range, so flat extrapolation isn't unreasonable.

A better approach is to factor in interest rate volatility.

- Option implied volatility comes from exchange rate volatility as well as the domestic and foreign interest rate volatilities.
- Interest rate volatilities are observable for longer maturities than exchange rate volatilities.

By using an FX model with stochastic interest rates calibrated to the full spectrum of interest rate options, one need only estimate a component of the full volatility. We used a stochastic volatility FX model with a normal short rate process for each interest rate.

## Market Conventions

To complete the picture, we need to populate our method with data. This is complicated by how option prices are quoted in the FX market:

- Quotes are in implied volatility, not in price.
- Quotes are for specific deltas, not strikes. The Black-Scholes delta of a call is monotonic in strike, so we can specify strikes in terms of their deltas.
- Delta of a call is  $100 \times$  Black-Scholes delta, using spot delta for maturity  $\leq 1$  year and forward delta afterward.
- Delta of a put is  $100 \times (-$  Black-Scholes delta  $)$ .
- Quotes are for ATM options, risk reversals (call vol - put vol for corresponding delta), butterflies (average of call vol and put vol - ATM vol), typically at 25 and 10 delta.
- Bids and asks are more complicated — one convention is that the RR bid/ask spread is actually the corresponding put volatility spread, and that the butterfly bid/ask spread is the call volatility spread.

## Market quotes to implied volatility

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We convert market quotes to implied volatilities as follows:

- Apply hole filling and interpolation methods above to the quoted mid vols.
- Construct spreads from quoted ATM spreads and from risk reversals and butterflies (as above).
- Interpolate spreads linearly in delta and  $\sqrt{t}$  with flat extrapolation, constrained to keeping bid, ask and spread nonnegative.
- Bid (ask) vols are interpolated mid vols minus (plus) interpolated spreads/2.

While interpolation of option vols seems like a simplification, in practice, it requires a substantial amount of sophistication.

## Barrier Options

Barrier options come in a tremendous variety of styles.

- One touch — Receive 1 euro if the EUR/USD FX rate passes 1.2 during the next month, otherwise 0.
- Knock out Barrier — Have a call on the FX rate at strike  $K$  as long as the barrier at  $B$  isn't touched.
- Similarly, no touches, double no touches, knock ins, double knock ins, ...
- Sequentials — Receive a call if the FX rate crosses  $B_1$  and then crosses  $B_2$ , or crosses  $B_2$  without hitting  $B_1$ , ...
- Stepped — Barrier levels change over time.

With this much variety, and so little quoted barrier pricing available, interpolation is infeasible — we need to choose a model.

## From Marginal to Joint Distribution

To choose a model, we should know what aspects of the model drive barrier pricing.

- Vanilla prices are driven by the distribution of  $X_{d/f}(T)$  for each  $T$  (the marginals of  $X$ ). In fact, knowing all vanilla prices tells us these distributions.
- One touches depend on the joint distributions of  $X(t)$  ( $0 \leq t \leq T$ ). More precisely, they're a function of  $\max_{t=0, T} X(t)$  or  $\min_{t=0, T} X(t)$ .
- Barriers depend on the joint distribution of  $X(T)$ ,  $\max_{t=0, T} X(t)$  and/or  $\min_{t=0, T} X(t)$ .

Whereas vanilla prices are driven by the marginals, barrier prices are driven by the joints.

# From Marginal to Joint Distribution

How to choose a model to use for barrier pricing?

- Choosing a model amounts to specifying the joints, and hence the barrier pricing.
- Choosing a model based on its fit to vanillas without considering its impact on the joints amounts to randomly selecting the joints.
- Drive the model — don't be driven by it.

How do we know what joints we want? In lieu of actual price data to calibrate our model to, we need something else.

- Historical analysis.
  - Does the model yield forward volatility structures similar to those observed over time?
  - Does the model hedge well?
- Take a view.

## Models for pricing barriers

If barriers don't price at Black-Scholes, where do they price? We've looked at a number of models. Some have been analyzed via hedging. Some by history.

- Black-Scholes
- Vanna-volga
- Semi-static hedging
- Stochastic vol
- Local vol
- Stochastic local vol
- Random risk reversal model

## Black-Scholes on barriers

Black-Scholes on barrier options has various issues:

- Doesn't match market quotes well — tends to under estimate the odds of hitting the barrier.
- Given the smile and term structure of volatility in vanillas, what vol should be used for barrier option pricing?

On the other hand, Black-Scholes is

- a good/necessary baseline starting point for valuation.
- easy to implement — all barriers on Europeans can be priced with formulas.

The (rarer) Americans and Bermudans can be priced with lattices or finite difference, if one either uses a dense enough grid, or takes care to place barriers and other singularities on grid points.

## Vanna-volga pricing

Vanna-volga pricing is another example of something that's conceptually simple, but becomes complex in practice. What started out as a trader's rule of thumb has grown into a number of variants, with a variety of explanations. One way to look at it is that it tries to correct for the impact of the smile on the Black-Scholes price.

- Define the Black-Scholes pricing error of a portfolio of vanilla options to be the market price minus the Black-Scholes price (using the ATM vol).
- "Correct" the exotic's Black-Scholes (with ATM vol) price by a weighted average of the Black-Scholes pricing errors for the delta-neutral straddle, the 25 delta butterfly, and the 25 delta risk reversal.
- The weights used are the positions needed for a portfolio of delta-neutral straddles, 25 delta butterflies and 25 delta risk reversals to have the same vega, vanna and volga as the option in question (under Black-Scholes at the ATM vol).
- Potentially rescale the weights according to specific features of the exotic option in question.

## Vanna-volga complications

In practice, vanna-volga pricing does a very good job of matching the market, but not before substantial tuning.

- Care must be taken in how the weights are rescaled — for basic one-touches and knock-outs, we reduce the adjustment by multiplying by the risk neutral probability (wrt the money market numeraire) of not hitting the barrier.
- To improve symmetry between working in foreign vs domestic currencies, we use the average of the survival probabilities under each corresponding risk neutral measure.
- Because this tends to yield too little adjustment when close to the barrier, we instead use  $S(p) = (1 + p)/2$ . In fact, we adjust the vega and volga components by this and the vanna component by  $p$ .
- Additional adjustments must be made for forward starting barriers, etc.
- Care must be used to avoid obvious arbitrages — the method must be used only on knock-ins and one-touches, and other barriers must be priced in terms of these and the vanilla prices.

## Digression — Vanna-volga and spreads

Generating bid/ask spreads for barrier options is problematic. While vanilla option bid/ask spreads can be computed by using separate bid and ask volatility surfaces, we can't use these for barrier bid/ask spreads.

- Consider a knock-out of a call under Black-Scholes.
- As the volatility increases, the chances of knocking out increase, reducing the value of the option.
- As the volatility increases, the value of call increases, increasing the value of the option.
- Depending on the relationship between the spot, barrier level, maturity, and strike, increasing vol can increase or decrease the value of the barrier option.

The Vanna-volga pricing method can be used to generate spreads as well, by attributing spreads (instead of pricing errors) to the vanna, vega and volga components.

## Semi-static hedging

Semi-static hedging involves establishing a portfolio of European options and a self-financing trading strategy to reproduce the payoff of a barrier option. Examples — Under Black-Scholes, a time  $T$  maturity one touch with level  $L$  can be replicated by the European option:

- $1_{X_T < L} \left( 1 + \left( \frac{X_T}{L} \right)^{2p} \right)$ , where
- $p = \frac{1}{2} - \frac{r_d - r_f}{\sigma^2}$ .

Buy a portfolio of time  $T$  vanillas that approximates the above payoff. Similarly, if the foreign and domestic interest rates are equal, this degenerates two digital puts, and  $1/L$  vanilla put, both struck at  $L$ . Price the one-touch as the cost of the hedge using market prices.

- Worked best in our hedging study.
- Haven't yet tried it in practice (vanna-volga is working well enough).

## Stochastic vol

We've also released a stochastic vol model (the Heston model):

- $dX = (r_d - r_f)Xdt + \sqrt{\nu}XdW_1$
- $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu}dW_2$
- $\langle dW_1, dW_2 \rangle = \rho$

Calibration approaches:

- Fit parameters to all vanilla prices.
  - Model has trouble fitting full surface — either misses short end or long end.
  - Barrier pricing is flawed by the poor fit of the model to vanilla prices.
- Fit parameters to option prices at barrier maturity.
  - Barrier pricing is better.
  - Model in some sense is being used to adjust the probability of hitting the barrier, which is thought to be higher under stochastic volatility than under Black-Scholes.
  - Lose benefits of pricing everything under one model.

We've made Heston available for both vanillas and barriers. For vanillas, we give up the short term fit, fitting it to the longer term structure component. For barriers, we are fitting to the vanillas at maturity.

## Stochastic vol numerical methods

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The Heston model has the nice property that there are many analytic results, leading to nice numerical methods.

- The characteristic function of  $X(T)$  is known.
- Vanillas can be priced by convolution.
- The convolution can be computed by adaptive integration or via FFTs.
- Barriers can be priced by finite difference using the model's pricing PDE.

## FFT based pricing

Let

$$\phi(u) = \mathbf{E}[e^{iu \log K}]$$

be the characteristic function of the log of the stock price at option expiration time. One way to approach FFT based pricing is to note that the Fourier transform of the  $e^{-\alpha k}$  dampened call price function (as a function of  $k = \log K$ ) can be written explicitly as a function of the characteristic function:

$$\omega(v) = \frac{e^{-rT\phi(v-(\alpha+1)i)}}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}$$

Then the option prices as a function of strike are given by inverting the Fourier transform of the above, and removing the damping term:

$$C(K) = \frac{e^{-\alpha \log K}}{2\pi} \int_{-\infty}^{\infty} e^{-iv \log K} \omega(v) dv$$

This integral can be computed directly via adaptive integration, but in that the above integral is essentially the Fourier transform of  $\omega(v)$ ,  $C(K)$  can be also be computed via the Fast Fourier Transform.

## FFT based pricing — pros and cons

The FFT method has a number of advantages:

- Relatively easy to set up.
- Gives prices for  $N$  strikes in  $O(N \log(N))$  (as opposed to  $O(N^2)$  for direct calculation of the convolution).
- Once set up, can easily test new, complicated models by just computing their characteristic function.

However,

- Not free to pick points — need to be careful, or take an extremely large number of points to get accurate values.
- FFT is not the Fourier transform — Errors creep in along the boundaries.
- Only applicable to Europeans and Bermudans — Not for barrier options.

## Local volatility

We're currently working on the release of a local volatility model:

- $dX = \mu X dt + \sigma(X_t, t) X dW$

Numerical Methods:

- Finite difference.

Calibration is subtle. There are two choices:

- Fit a functional form of the local volatility.
  - Won't yield an exact fit to the entire vol surface, but it's easy to hit all the quotes.
  - Computationally intensive.
- Differentiate the implied volatility surface (via the Dupire equations) to directly compute the local volatility.
  - Vol surface must be twice differentiable in strike, differentiable in time, and arbitrage free.
  - Can lead to oscillatory local vol surfaces.
  - Will more closely hug entire (interpreted) implied surface, but errors can accumulate, leading to some slippage and missing of quoted option prices.

We're currently leaning towards the second approach.

## Stochastic local vol

Trying for the best of both worlds, consider

- $dX = (r_d - r_f)Xdt + \sigma_s(X_t, t)\sqrt{\nu}XdW_1$
- $d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu}dW_2$
- $\langle dW_1, dW_2 \rangle = \rho$

Numerical methods:

- Finite difference.

Calibration

- Fix parameters (except for  $\xi$  and  $\sigma_s$ ).
- Select  $\xi$ .
- Find  $\sigma_s$  to match the observed vanilla prices.
- $\xi$  then parameterizes a family of models fitting the vanillas.
- $\xi$  controls the joint distributions without disturbing the vanilla prices, allowing (we think) some direct control over barrier pricing.

## Random risk reversal

### Random risk reversal model

- Diffusion with up and down jumps (to give skew).
- Time-changed to inject autocorrelation necessary to make skew persistent in time.

### Features

- Captures slowly changing risk reversal prices.
- Fits vol surface better than Heston (but also more parameters).
- Fits are also surprisingly good when parameters are set via historical fitting and state variables are selected to match current market.

### Numerical methods

- Model has a closed form for the characteristic function.
- Vanillas can be priced can price vanillas via convolution (or FFT).
- Barrier pricing is difficult.

We've mostly shelved work on this while pursuing the stochastic local volatility model.

## Model smackdown

If a model is pricing barrier options properly, then the hedges that the model produces should do a good job of hedging the barrier option. This leads to testing models by simulating their hedging behavior on historical data. This experiment doesn't even need barrier option prices — the model price can be used.

Tested:

- Black-Scholes
- Heston (with global calibration)
- Semi-static hedging method.

## Hedging analysis results

### Historical hedging performance results:

- Semi-static hedging performed best by far.
- Heston with delta/vega hedge comes in second.
- Black-Scholes with delta hedge is by far the worst.
- Black-Scholes with delta/vega hedge does almost as good as Heston.

### Interpretation:

- That native Heston (and Black-Scholes with added vega hedge) work much better than delta hedging under Black-Scholes is evidence of stochastic volatility.
- That the semi-static hedge works better than either is evidence of additional risks (on top of stochastic volatility) that are hedged by the semi-static hedge but not by the other hedges.

## Summary

We've covered the following points.

- Mathematical framework for pricing FX options.
  - Foreign and domestic bonds and exchange rates.
  - Forwards and options.
  - Stochastic rates? Why?
- Behavior and pricing of vanillas.
  - Market far from Black-Scholes.
  - Interpolation and extrapolation.
  - Business time = time dependent volatility.
  - Bid/ask spreads.
- Behavior and pricing of barriers.
  - Drive the model, don't be driven by it.
  - Barrier pricing models:
    - Black-Scholes
    - Vanna-volga
    - Semi-static hedging
    - Stochastic vol
    - Local vol
    - Stochastic local vol
    - Random risk reversal.
  - Model hedging performance.

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