Valuing Options on Dividend–Paying Stocks

If you're using the wrong model to value convertible bonds or Leaps on dividend-paying stocks, you may be selling options too cheaply

Many investors who are familiar with the standard treatments for pricing equity options may not realize that those approaches usually focus on short-dated options. But problems arise when the standard methods are used in the context of long-dated options. Of special concern is the pricing of equity options that have maturities extending out for several years or more. The conversion option in a convertible bond is an example of such a long-dated option. Other examples include the longer-term Leaps (long-term equity anticipation securities), which are equity options listed on the Chicago Board Options Exchange with expirations of up to three years. Investors should pay special attention to the ways dividend payments on underlying common stock affect option pricing.

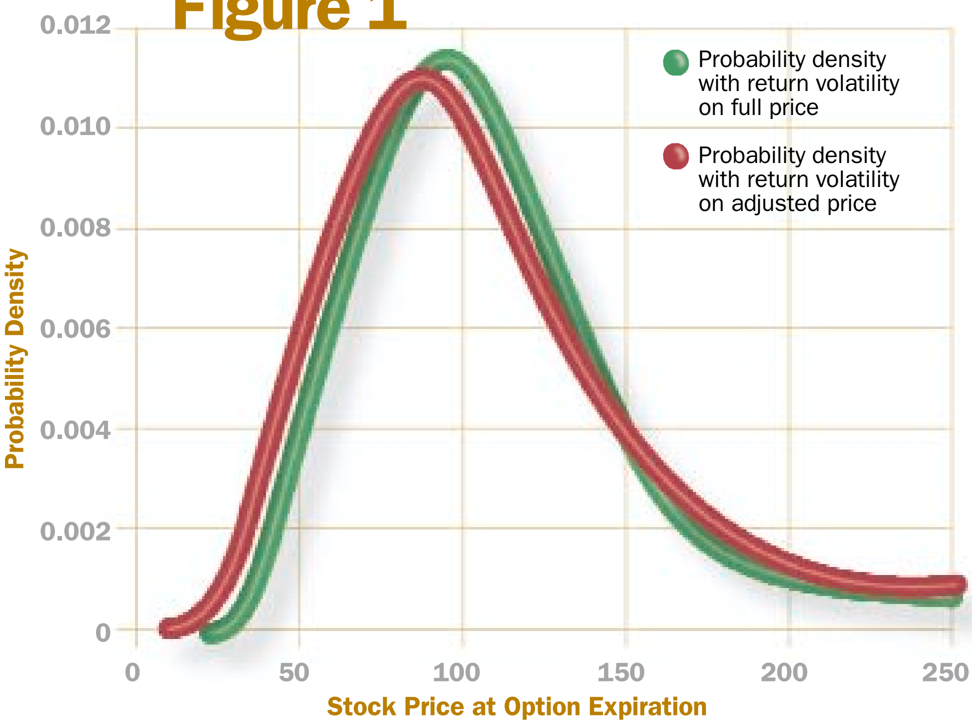
The classic Black-Scholes formula gives the value for a European call on a stock that pays no dividends during the life of the option. For stocks that pay no dividends, it's not economical to exercise a call option before option expiration, so that same formula also is valid for an American call option.

Even 20 years ago, researchers realized that many stocks pay periodic dividends. That prompted Richard Roll of the University of California, Los Angeles, to develop an extension of the Black-Scholes formula to treat the case of an American exercise call option when the underlying security pays known dividend payments before option expiry.[1] Roll used the formula of Robert Geske, also of the University of California, Los Angeles, for a compound option, and Geske, too, developed some formulas in this area, including how to extend the formula to handle several escrowed dividend payments.[2]

In the Roll and Geske formulas, the underlying stochastic variable is taken to be the adjusted stock price, which is defined as current stock price less present value of all dividends occurring up to option expiration. The idea is that the unknown evolution of a future stock price involves only that part of the stock price that is distinct from the known—and escrowed—dividends.

What Roll (page 256), Geske (footnote, page 378), and other authors stress, repeatedly, is that in computations for adjusted price, only dividends that are known with certainty are to be included. Roll comments that "two or more known successive

Figure 1



Comparison of probability density functions obtained with two different treatments of volatility. The graphs refer to a stock with initial price 100; constant annual dividends of five at years one, two, and three; σ equal to 20 percent; r equal to 5 percent; and a horizon of three and a half years

Figure 2

Strike	Call Premium If Volatility Is on Full Price	Call Premium If Volatility Is on Adjusted Price	Absolute Difference in Premiums	Percentage of Difference in Premiums
100	15.07	13.93	1.14	7.56
105	13.27	12.12	1.15	8.66
110	11.66	10.52	1.14	9.79
115	10.23	9.11	1.12	10.95
120	8.96	7.87	1.09	12.12
125	7.84	6.79	1.04	13.32
130	6.84	5.85	0.99	14.52

Comparison of option premiums obtained with two different treatments of volatility. The table refers to a call option on a stock with initial price 100; constant annual dividends of five at years one, two, and three; σ equal to 20 percent; r equal to 5 percent; and option expiry of three and a half years

dividends are rare for common stocks”; Geske agrees with that view and adds that for very long-dated options, the adjusted price can be approximately zero, which is of course not a very useful starting point.

It’s better to take the stochastic variable to be the full stock price and not the adjusted stock price. That implies that the modeled stock price evolution will exhibit jumps at ex-dividend dates, as is observed in the real world. More important, the effect of the volatility parameter ($\boldsymbol{\sigma}$) will be on the return on the full price. We believe this is the correct interpretation of the volatility parameter. It is the return on the full price that’s observed in the marketplace, and it is the volatility of the full-price return that’s quoted in short-dated options when there’s no ex-dividend date before option expiry. In addition, historical volatility is computed based on quoted prices and not on quoted prices less anticipated dividends—with the obvious proviso that on historical ex-dividend dates one would make adjustments for known dividends. We also believe that when adjusted-price models are invoked, users interchange full-price and adjusted-price return volatility—with no thought that there might be a difference. So it seems prudent to use a model based on full-price return volatility, which is what is typically quoted.

Looking more closely at the interpretation of volatility, we’ll focus on European exercise options. The substance of the discussion on how to treat volatility carries through to the case of American exercise as well.

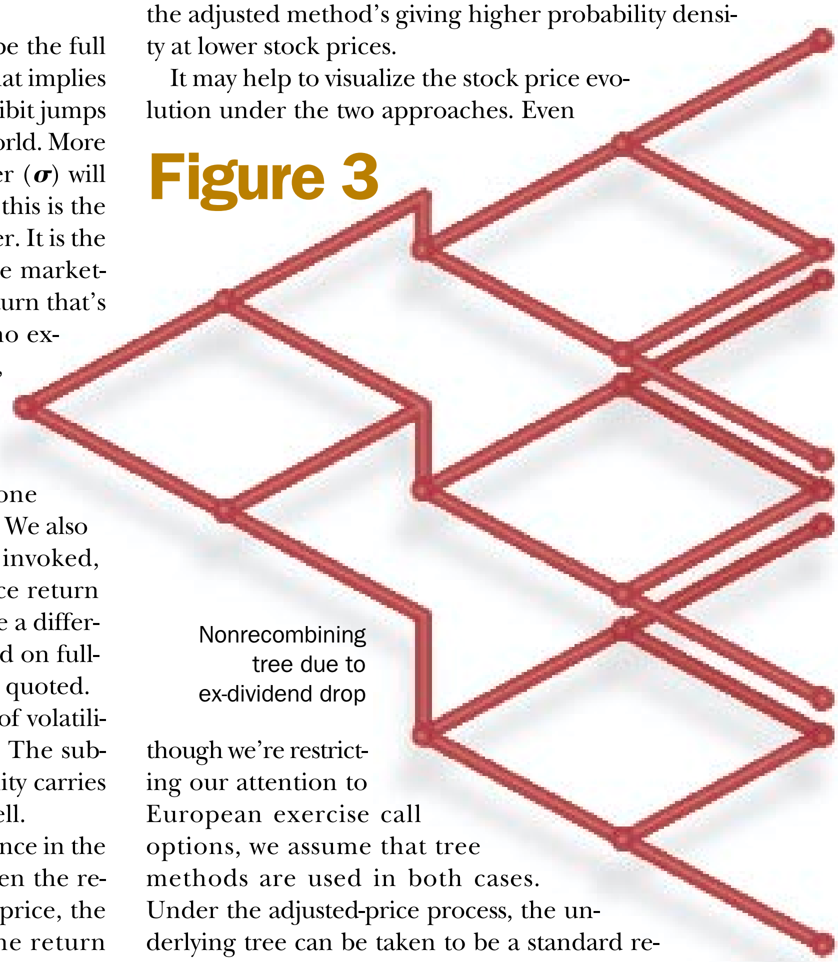
The first thing to note is that there’s a difference in the results of the two approaches. In general, when the return volatility is interpreted to be on the full price, the premium computed is higher than when the return volatility is interpreted to be on the adjusted price. We can see that result in two different ways. Considering a fixed horizon—as we would in the case of a European exercise option—the resulting distribution of stock prices differs in the two approaches.

In figure 1 we compare density functions for the stock price at a specific horizon. In figure 2 we show the value of European call options computed by means of the two different treatments of volatility.

Figure 2 contains premiums calculated by two methods: the usual adjusted-price method, described in the introduction, and the full-price method advocated here. The example considers a stock with initial price 100; constant annual dividend of five at years one, two, and three; and a constant interest rate (r) at 5 percent (continuous compounding). The option is a call with expiry of three and a half years and with premiums calculated for a variety of strikes. The results show an increasing relative difference for higher strikes, with the full-price method consistently giving a higher premium. Those results are indicated qualitatively in figure 1, which shows the adjusted method’s giving higher probability density at lower stock prices.

It may help to visualize the stock price evolution under the two approaches. Even though we’re restricting our attention to European exercise call options, we assume that tree methods are used in both cases. Under the adjusted-price process, the underlying tree can be taken to be a standard recombining binary tree, as introduced by John Cox of

Figure 3



Nonrecombining tree due to ex-dividend drop

Massachusetts Institute of Technology, Stephen Ross of Yale University, and Mark Rubinstein of the University of California, Berkeley. Under the full-price process, the tree is no longer recombining.

Figure 3 shows a nonrecombining tree. Note that the drop is not the same size at each node, because the tree is plotted in the log scale. The regularity of the recombining tree makes for easier algorithm development, but the simplification comes at the cost of selecting a stochastic process that may be less appropriate for option valuation.

A criticism that can be leveled at the adjusted-price method is that it gives rise to discontinuities. Discontinuities are telltale signs that ad hoc procedures have been used in a model to remove a difficulty. In this case, the difficulty confronted is the increased complexity inherent in dealing with a nonrecombining tree when discrete dividends are modeled directly. The ad hoc procedure is to adjust the stock price by the discounted value of any dividends that occur between settle and option expiration.

Figure 4

Option premiums as a function of the option expiration date. The graphs refer to a stock with initial price 100, one discrete dividend of 10 at year one, σ equal to 20 percent, and r equal to 5 percent. For option expiry shorter than one year, the strike is 100; for option expiry greater than one year, the strike is 90

In figure 4 we compare option premiums as a function of option expiration date. The most alarming feature is the discontinuity of the premium in the adjusted-price method when the option expiration crosses the ex-dividend date. The graphs refer to a stock with initial price 100, one discrete dividend of 10 at year one, σ equal to 20 percent, and r equal to 5 percent. For option expiry shorter than one year, strike is 100; for option expiry greater than one year, strike is 90. Such adjustment should lead to a continuous premium for the following reason. The payout on a European call option with strike K and expiry just before an ex-dividend date would be identical to the payout on a European call option with strike K minus D and expiry just after an ex-dividend date, assuming the ex-dividend drop is D; see the appendix for mathematical formulation of this statement. If the discontinuity were in force in over-the-counter contracts, one would exploit it by writing the shorter-maturity call, purchasing the longer-maturity call that is an almost-perfect hedge, and pocketing the substantial difference. The option premiums converge as expiration approaches.

To be fair, both of the option valuation approaches discussed here can be made to appear unreasonable in extreme cases. Consider the case of high-dividend projections and significant volatility. In such a situation, the full-price method needs some adjustment; otherwise, the model leads to the possibility of negative stock prices in the future. Similarly, the adjusted-price process shares the problem of assuming the constant dividend will be paid under all future stock price scenarios—even those wherein the stock price drops significantly. Such problems are not shared by a model that uses a proportional dividend rate. Unfortunately, in modest volatility and dividend cases, a proportional dividend rate model does not capture the important, discrete nature of the dividends.

Again, when you value stock options—or the option to convert in a convertible bond—the stochastic process you choose for the stock should be on the full stock price and not the adjusted price. That's because most investors quote volatility on the full price, and that volatility is then comparable across options of different maturities; the volatility on an adjusted price depends on the option maturity in a discontinuous way. And option premiums under the adjusted-price method lead to theoretical discontinuities that would appear to lead to arbitrage opportunities. In general, for the same quoted volatility, effective volatility is higher in the full-price method than in the adjusted-price method, which leads to higher—theoretical—option premiums in this method vis-à-vis the adjusted-price method, especially as the option expiration extends. This can have clearly noticeable effects on securities with embedded equity options such as convertible bonds.

Appendix

Two criteria should be met by a reasonable pricing model for options on a stock with discrete dividends. They are the consequences of two reasonable assumptions, or principles, about real-world market behavior.

- *Principle 1: Independence.* The real-world stochastic process for a stock's price is not influenced by a nonrelated party's issuance of a derivative security whose underlying security is that stock.
- *Principle 2: Ex-dividend drop.* The expected drop in a stock price in the real world—due to the stock's going ex-dividend—is αD, where D is the size of the dividend and α is a constant—usually, one.[2]

The two principles have been somewhat carefully worded, but the intuition is clear. If we had wanted to consider only stock options, we could have simplified the wording somewhat.

Nonrelated party The expression *nonrelated party* rules out changes in capital structure. If the issuer is the company itself, then the derivative security—for example, warrants—would change the firm's capital structure, which could very well affect the stochastic behavior of individual components within the capital structure. For a discussion of warrants versus stock options, see [3].

The principles mentioned here would likely also apply to the modeling world. But that statement needs clarification. Any model is valid if it returns the right answer given the real-world assumptions being made. For example, from Black-Scholes theory we know that the investor's risk preferences do not appear in the partial differential equation for the price of the stock option and they therefore will not appear in the solution. Because of that, when developing a model for an option price, one can use any choice of investor-risk preferences—and it won't change the answer. The usual choice is to take a risk-neutral investor, because for that choice we can write down immediately the correct discounting rate: the risk-free rate. But the choice of risk-neutral preferences is not required to get the unique answer of the option price in a Black-Scholes economy.

The α factor The price drop when a stock goes ex-dividend may be less than the dividend depending on tax-related issues. We generally take α to equal one. See [4] for a discussion of this issue.

Two criteria must be met by a proposed model:

- *Criterion 1: Volatility.* The volatility of the model should be stated in terms of the full stock price and not the adjusted price. For the special case of the assumption of a lognormal stock price process, the assumption is that the volatility of the full price is constant. It is the volatility of the fall price that is observable, that usually is quoted, and that is used for, say, options whose life is within a dividend-free period for the underlying security. Using a different definition of *volatility* for different derivative securities violates the spirit of the first principle and can lead to inconsistencies.
- *Criterion 2: Density.* Let t_d be an ex-dividend date—that is, the time at which the stock's price drops by an amount D_{t_d}—and let t_d^- be the moment just before t_d. Similarly, let t_d^+ be the moment just after t_d. Then the following property holds for $S(t)$—the real-world stock price process:

$$\text{Prob}(A < S(t_d^-) < B) = \text{Prob}(A < S(t_d^+) + D_{t_d} < B) \quad (1)$$

- *Corollary: Crossing ex-dividend dates.* Let $S(t)$ be the stock price process for a stock with a single known dividend at time t_d of size D. Let $c(K,t,T)$ denote the price at time t of a European call on S with expiry T and strike K. Then the following relationship holds:

$$c(K,t,t_d^+) = c(K+D,t,t_d^-) \quad (2)$$

Proof. For $t_1 < t_2$, let $d\boldsymbol{\mu}(S(t_1)|S(t_2))$ denote the density of $S(t_2)$ conditional on the value of S at time t_1 being $S(t_1)$. Then

$$\begin{aligned} c(K,t,t_d^+) &= \textstyle\int(S(t_d^+) - K)^+ d\boldsymbol{\mu}(S(t_d^+)|S(t)) \\ &= \textstyle\int(S(t_d^-) - D - K)^+ d\boldsymbol{\mu}(S(t_d^-)|S(t)) \\ &= \textstyle\int_{K+D}^{\infty}(S(t_d^-) - (K+D))\, d\boldsymbol{\mu}(S(t_d^-)|S(t)) \\ &= c(K+D,t,t_d^-) \qquad \Delta \end{aligned} \quad (3)$$

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[4]Giovanni Barone-Adesi and Robert E. Whaley. "The Valuation of American Call Options and the Expected Ex-Dividend Stock Price Decline," *Journal of Financial Economics,* vol. 17(l) (1986): 91–111.