Global Derivatives Trading and Risk Management

A MULTI-FACTOR MARKOVIAN HJM MODEL FOR PRICING EXOTIC INTEREST RATE DERIVATIVES

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May 20-22, 2008, Paris

PRESENTATION

- **AREA:** INTEREST RATE DERIVATIVES VALUATION
- **FRAMEWORK :** MARKOVIAN HJM FRAMEWORK
- **APPROACH:** STATE VARIABLE INDUCTION / EVOLUTION
- **TASK:** EFFICIENT DERIVATIVES PRICING

- **PLAN:**
  - Review multi-factor Markovian HJM framework
  - Introduce parsimonious multi-factor volatility specifications
  - Elaborate generic lattice-building approach (Numerical Results)
  - Conclusions
  - References

General Multi-Factor HJM Framework

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)\cdot dW(t) = \mu(t,T)dt + \sum_{i=1}^{N}\sigma^{i}(t,T)dW_{i}(t)$$

$$\mu(t,T) = \sigma(t,T)\cdot\int_{t}^{T}\sigma(t,s)ds = \sum_{i=1}^{N}\sigma^{i}(t,T)\int_{t}^{T}\sigma^{i}(t,s)ds$$

$$P(t,T) = e^{-\int_{t}^{T} f(t,s)ds}$$

Advantages of Markovian HJM Framework

	Markovian wrt Finite Set of State Variables	Term Structure - Input of the Model (automatically recovered)
Short Rate Models	Yes	No
General HJM Models	No	Yes
Markovian HJM Models	Yes	Yes

Possibility for Efficient Lattice Building

Analytical State Discount Factors

Efficient Pricing American/Bermudan Derivatives

Markovian HJM Framework: Structure

Volatility: $\sigma(t,T)$

State Variables

Evolution

Forward Rates $f(t,T)$

Short Rate $r(t)$

Discount Factors $P(t,T)$

General Markovian HJM Framework: Separable Volatility I

Volatility: $$\sigma(t,T) = h(t)g(T)$$

$$g(T) = e^{-\int_0^T \kappa(s)ds}$$

$g(T)$	**is *N*-dimensional vector**
$h(t)$	**is (*N* x *N*)-dimensional matrix**
$\kappa = \kappa(s)$	**is a deterministic function of time**

State Variables:

$$X(t) = \int_0^t Y(s)g(s)ds + \int_0^t h^*(u)dW(u)$$

$$Y(t) = \int_0^t h^*(u)h(u)du$$

Evolution:

$$dX(t) = Y(t)g(t)dt + h^*(t)dW(t)$$

$$dY(t) = h^*(t)h(t)dt$$

General Markovian HJM Framework: Separable Volatility II

Forward Rates:

$$f(t,T) = f(0,T) + \frac{1}{2}[g^*(T)Y(t)G(t,T) + G^*(t,T)Y(t)g(T)] + g^*(T)X(t)$$

Short Rate:

$$r(t) = f(t,t) = f(0,t) + g^*(t)X(t)$$

$$G(t,v) = \int_t^v g(s)ds$$

Discount Factors:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\frac{1}{2}G^*(t,T)Y(t)G(t,T) - G^*(t,T)X(t)\right]$$

General Markovian HJM Framework I: Andreasen (2005)

State Variables:

$$\tilde{X}(t) = I_{g(t)} X(t)$$

$$\tilde{Y}(t) = I_{g(t)} Y(t) I_{g(t)}$$

$$I_a = diag(a_1, \ldots a_N)$$

Evolution:

$$d\tilde{X}(t) = \tilde{Y}(t) J dt - I_{\kappa(t)} \tilde{X}(t) + \eta(t) dW(t)$$

$$d\tilde{Y}(t) = \eta(t)\eta^*(t) - I_{\kappa(t)} Y(t) - Y(t) I_{\kappa(t)}$$

$$J = (1, \ldots 1)^*$$

$$\eta(t) = I_{g(t)} h^*(t)$$

General Markovian HJM Framework II: Andreasen (2005)

Forward Rates:

$$f(t,T) = f(0,T) + \frac{1}{2}[g^*(t,T)\tilde{Y}(t)\tilde{G}(t,T) + \tilde{G}^*(t,T)\tilde{Y}(t)g(t,T)] + g^*(t,T)\tilde{X}(t)$$

$$g(t,T) = e^{-\int_t^T \kappa(s)ds}$$

$$\tilde{G}(t,T) = \int_t^T g(t,s)ds$$

Short Rate:

$$r(t) = f(t,t) = f(0,t) + J\ \tilde{X}(t) = f(0,t) + \sum_{i=1}^{N} \tilde{X}_i(t)$$

Discount Factors:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\frac{1}{2}\tilde{G}^*(t,T)\tilde{Y}(t)\tilde{G}(t,T) - \tilde{G}^*(t,T)\tilde{X}(t) \right]$$

Example 1. RS Model: 1-Factor 2-State Variable Model

Volatility: $\sigma_f(t,T) = \sigma_f(t,t)\cdot k(t,T)$ ← $k(t,T) = e^{-b(t,T)}, \quad b(t,T) = \int_t^T \kappa(s)\cdot ds$

State Variable: $\phi(t) = \int_0^t \sigma_f^2(u,t)\cdot du = \int_0^t \sigma_f^2(u,u)\cdot k^2(u,t)\cdot du$

$$dr(t) = \mu(r,\phi,t)\,dt + \sigma_f(t,t)\cdot dz(t)$$

$$d\phi(t) = [\sigma_f^2(t,t) - 2\kappa(t)\cdot\phi(t)]dt, \quad \phi(0) = 0$$

Evolution: $\mu(r,\phi,t) = \kappa(t)\cdot(f(0,t) - r(t)) + \phi(t) + f_t(0,t)$

Discount Factors:

$\beta(t,T) = \int_t^T k(t,s)\cdot ds$ → $P(t,T) = \frac{P(T)}{P(t)}\exp\left[-\beta(t,T)\cdot(r(t) - f(0,t)) - \frac{1}{2}\beta^2(t,T)\phi(t))\right]$

Example 2. RC Model: 1-Factor 3-State Variable Gaussian Model: Reproducing Volatility Hump

Volatility:

$$\sigma_f(t,T) = (a + c(T-t)) \cdot e^{-k(T-t)} + b$$

State Variables:

$$W_0(t) = \int_0^t dw(v), \quad W_1(t) = \int_0^t e^{-\kappa(t-v)} dw(v), \quad W_2(t) = \int_0^t (t-v) \cdot e^{-\kappa(t-v)} dw(v)$$

Evolution:

$$dW_1(t) = -\kappa W_1(t)dt + dW_0(t), \quad dW_2(t) = (W_1(t) - \kappa W_2(t))dt$$

Forward Rates:

$$f(t,T) = f(0,T) + h(t,T) + \sum_{i=0}^{2} d_i(t,T) \cdot W_i(t)$$

$$d_0(t,T) = b, \quad d_1(t,T) = (a + c(T-t))e^{-\kappa(T-t)}, \quad d_2(t,T) = ce^{-\kappa(T-t)}$$

Short Rate:

$$r(t) = f(t,t) = f(0,t) + \theta(t) + b \cdot W_0(t) + a \cdot W_1(t) + c \cdot W_2(t)$$

Example 3. Markovian HJM Model: Kramin et al. (2008)
M-Factor - Constant Mean Reversions

$$\sigma_j(t,T) = \sum_{i=1}^{N_j} h_{ij}(t) \cdot e^{-\beta_{ij}(T-t)}$$

$$\beta_{ij} = const, \quad i = \overline{1, N_j},\ j = \overline{1, M}$$

1-Factor 2-Term 5-State Variable Model:

$$\sigma(t,T) = h_1 \cdot e^{-\beta_1(T-t)} + h_2 \cdot e^{-\beta_2(T-t)}$$

2-Factor 2-Term 10-State Variable Model:

$$\sigma_1(t,T) = h_{11} \cdot e^{-\beta_{11}(T-t)} + h_{21} \cdot e^{-\beta_{21}(T-t)}$$

$$\sigma_2(t,T) = h_{12} \cdot e^{-\beta_{12}(T-t)} + h_{22} \cdot e^{-\beta_{22}(T-t)}$$

2-Factor 1-Term 4-State Variable Model:

$$\sigma_1(t,T) = h_1 \cdot e^{\kappa_1(T-t)}$$

$$\sigma_2(t,T) = h_2 \cdot e^{-\kappa_2(T-t)}$$

M-Factor 1-Term 2M-State Variable Model:

$$\sigma_j(t,T) = h_j(t) e^{-\kappa_j(T-t)}$$

$$\kappa_j = const, \quad N_j = 1, \quad j = \overline{1, M}$$

Choice of h-functions: Skew Specification

CEV Approach:

$$h_j(t) = \sigma_j(t) R_j^{\gamma_j}(t) R_j^{1-\gamma_j}(0)$$

Shifted-Lognormal Approach:

$$h_j(t) = \sigma_j(t)[\gamma_j R_j(t) + (1-\gamma_j) R_j(0)]$$

Short, Forward, Swap Rates Based h-functions:

$$R_j(t) = \{r(t),\ f(t, T_j), S_{n_j, m_j}(t)\}$$

h-functions examples:

LRS One-factor model:

$$h(t) = \sigma \cdot [r(t)]^{\gamma}$$

IK Two-factor model:

$$h_1(t) = \sigma_1 r^{\alpha}(t),\ \ h_2(t) = \sigma_2 r^{\beta}(t)$$

Andreasen One-factor model:

$$h(t) = \zeta(t, x(t)) = \lambda(t) x^{\alpha}(t) x^{1-\alpha}(0),$$
$$x(t) = \{S_{k,n}(t), t \in (T_{k-1}, T_k]\}$$

Example: 2-Factor 4-State Variable HJM Model (IK) I

Volatility: $\sigma_1(t,T) = \sigma_1 r^{\alpha}(t)$ $\quad$ $\sigma_2(t,T) = \sigma_2 r^{\beta}(t) e^{-\kappa(T-t)}$

State Variables:

$$Q_1(t) = \sigma_1^2 \int_0^t r^{2\alpha}(\tau) d\tau$$

$$Q_2(t) = \sigma_2^2 \int_0^t r^{2\beta}(\tau) \frac{e^{-2\kappa(t-\tau)}}{\kappa} d\tau$$

$$L_1(t) = \sigma_1^2 \int_0^t r^{2\alpha}(\tau)(t-\tau) d\tau + \sigma_1 r^{\alpha}(\tau) dz_1(\tau)$$

$$L_2(t) = \sigma_2^2 \int_0^t r^{2\beta}(\tau) \frac{e^{-\kappa(t-\tau)}}{\kappa} d\tau + \sigma_2 r^{\beta}(\tau) e^{-\kappa(t-\tau)} dz_2(\tau)$$

Evolution:

$$dQ_1(t) = \sigma_1^2 r^{2\alpha}(t) dt$$

$$dQ_2(t) = -2\kappa Q_2(t) dt + \sigma_2^2 r^{2\beta}(t) \frac{1}{\kappa} dt$$

$$dL_1(t) = Q_1(t) dt + \sigma_1 r^{\alpha}(t) dz_1(t)$$

$$dL_2(t) = -\kappa L_2(t) dt + \sigma_2^2 r^{2\beta}(t) \frac{1}{\kappa} dt + \sigma_2 r^{\beta}(t) dz_2(t)$$

Example: 2-Factor 4-State Variable HJM Model (IK) II

Forward Rates:

$$f(t,T) = f(0,T) + Q_1(t)(T-t) - Q_2(t)e^{-2\kappa(T-t)} + L_1(t) + L_2(t)e^{-\kappa(T-t)}$$

Short Rate:

$$r(t) = f(0,t) + L_1(t) + L_2(t) - Q_2(t)$$

Discount Factors:

$$P(t,T) = \frac{P(T)}{P(t)} \exp\left[\frac{1}{2} Q_1(t)(T-t)^2 + Q_2(t)\frac{1-e^{-2\kappa(T-t)}}{2\kappa} - L_1(t)(T-t) - L_2(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} \right]$$

Markovian HJM Model: Number of state Variables

Ritchken and Sankarasubramanian - RS(1995):	**2**	**One Factor**
Ritchken and Chuang - RC(1999):	**3**	**One Factor**
Inui and Kijima - IK(1998):	**2M**	**M Factors**
Andreasen (2005):	$\frac{M^2 + 3M}{2}$	**M Factors**
Kramin, Nandi, Shulman (2008):	$\frac{1}{2}\sum_{j=1}^{M}\left[N_j^2 + 3N_j\right]$	**M Factors**
Kramin (2008):	**2M**	**M Factors**

LRS Model: CEV Approach, Nelson Transformation

LRS Approach: Nelson Transformation (Change of Variables):

$$\sigma_f(t,t) = \sigma \cdot [r(t)]^\gamma \longrightarrow Y(t) = \int \frac{dr(t)}{\sigma_f(t,t)}$$

$$d\phi(t) = [\sigma_f^2(t,t) - 2\kappa(t) \cdot \phi(t)]dt, \quad \phi(0) = 0$$

$$dY(t) = m(Y(t), \phi(t), t) \cdot dt + dz(t)$$

$$\uparrow$$

$$m(Y(t), \phi(t), t) = \left[\frac{\partial Y(t)}{\partial t} + \mu(r(t), \phi(t), t) \cdot \frac{\partial Y(t)}{\partial r(t)} + \frac{1}{2} \cdot \sigma^2(r(t), \phi(t), t) \cdot \frac{\partial^2 Y(t)}{\partial r(t)^2} \right]$$

LRS Model: Various Distributional Assumptions

Log-normal model

$$\gamma = 1$$

$$Y(t) = \frac{\ln(r(t))}{\sigma}$$

$$m(Y,\phi,t) = (\kappa(t)\cdot(f(0,t) - e^{\sigma\cdot Y(t)}) + \phi(t) + f_t(0,t))e^{-\sigma\cdot Y(t)}\frac{1}{\sigma} - \frac{\sigma}{2}$$

Square-root model

$$\gamma = 0.5$$

$$Y(t) = \frac{2\sqrt{r(t)}}{\sigma}$$

$$m(Y,\phi,t) = \left[\kappa(t)\cdot\left(f(0,t) - \frac{\sigma^2\cdot Y^2(t)}{4}\right) + \phi(t) + f_t(0,t))\cdot\frac{2}{\sigma^2} - \frac{1}{2}\right]\cdot\frac{1}{Y(t)}$$

Normal (HW) model

$$\gamma = 0$$

$$Y(t) = \frac{r(t)}{\sigma}$$

$$m(r,\phi,t) = \kappa(t)\cdot(f(0,t) - \sigma Y(t)) + \phi(t) + f_t(0,t)$$

$$\phi(t) = \frac{\sigma^2}{2\kappa}\cdot(1 - e^{-2\kappa t})$$

$$\beta(t,T) = \frac{1}{\kappa}\cdot(1 - e^{-\kappa(T-t)})$$

Binomial Lattice Implementation: LRS and RC Models

LRS Model

$$Y_0 \rightarrow \begin{cases} Y_u = Y_0 + (J+1)\sqrt{\Delta t} \\ Y_d = Y_0 + (J-1)\sqrt{\Delta t} \end{cases}$$

$$p = \frac{m(Y_0, \phi_0, t_0)\sqrt{\Delta t} - (J-1)}{2}$$

RC Model

$$W_0(t_0) = w_0 \rightarrow \begin{cases} W_0^u = w_0 + \sqrt{\Delta t} \\ W_0^d = w_0 - \sqrt{\Delta t} \end{cases}$$

$$p = \frac{1}{2}$$

$$W_1(t_0) = w_1 \rightarrow \begin{cases} W_1^u = (1 - \kappa \cdot \Delta t)w_1 + \sqrt{\Delta t} \\ W_1^d = (1 - \kappa \cdot \Delta t)w_1 - \sqrt{\Delta t} \end{cases}$$

$$p = \frac{1}{2}$$

$$W_2(t_0) = w_2 \rightarrow W_2(t_0 + \Delta t) = (1 - \kappa \cdot \Delta t)w_2 + w_1$$

Markovian HJM Model Lattice Implementation Problem

- **At any node the total number of distinct permissible values for auxiliary state variables is the total number of unique lattice paths leading to this node**

- **To capture the distribution of the auxiliary state variables one can take into account all permissible values for auxiliary state variables at every node, which is not computationally feasible: the number of permissible values grows exponentially with the number of time steps**

- **This problem can be solved using a special numerical lattice technique that is practically feasible and does not cause a loss of information, which distorts the distribution of the auxiliary state variables**

One-Factor HJM Lattice: LRS, RC, LR, KKDY Models

LRS(1995), RC(1999): Distribution of the auxiliary state variables

- not to track all the values for this state variable
- identify two paths from the origin to each node: max and min values for every aux. state variable
- partition obtained interval into a finite number of points, at which values are calculated

***L*-dimensional matrix explains distribution of *L* state variables at every node**
Adding a new state variable ⟹ Increase in dimensionality of the problem

LR (2006): Reconnecting lattice

Vector of values at every node ⟹ Conditional expectation of corresponding aux. state variable
Apply an iterative quadratic interpolation/extrapolation to implement backward recursion.

Kramin et al. (2005): Induction Method

- **Forward Induction for conditional expectations for all state variables**
- **Simple backward induction using conditional expectations for valuation**

No adjustments and interpolations
No significant loss of information

Gyongy Theorem

Stochastic process (SDE): $dX(t) = \alpha(t)dt + \beta(t)dW(t)$

$\alpha = \alpha(t), \beta = \beta(t)$ **adapted bounded stochastic processes**

Define:

$$a(t,x) = E(\alpha(t) \mid X(t) = x)$$
$$\mathrm{b}^2(t,x) = E(\beta^2(t) \mid X(t) = x)$$

Local Volatility process (SDE): $dY(t) = a(t,Y(t))dt + b(t,Y(t))dW(t),\ \ Y(0) = X(0)$

Result: Local Volatility SDE admits a weak solution Y that has the same (one-dimensional) distribution as the solution of the original SDE X

Application of Gyongy Theorem to RS Model

Stochastic process (SDE):

$$dr(t) = [\kappa(t)\cdot(f(0,t) - r(t)) + \phi(t) + f_t(0,t)]dt + \sigma(t,r(t))\cdot\ dz(t)$$

$$d\phi(t) = [\sigma^2(t,r(t)) - 2\kappa(t)\cdot\phi(t)]dt, \quad \phi(0) = 0$$

Conditional Expectation of Secondary Variable $\bar{\phi}(t,r) = E(\phi(t)\,|\,r(t) = r)$

Simplified stochastic process (SDE):

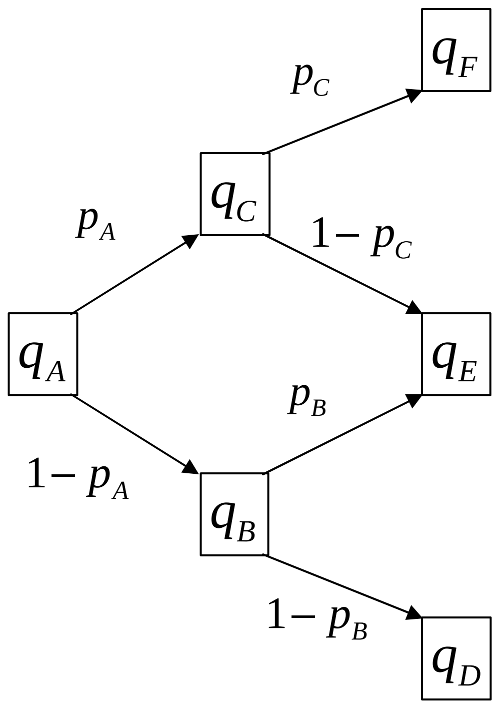
$$d\bar{r}(t) = [\kappa(t)\cdot(f(0,t) - r(t)) + \bar{\phi}(t,r(t)) + f_t(0,t)]dt + \sigma(t,r(t))\cdot\ dz(t)$$

$$d\bar{\phi}(t,r(t)) = [\sigma^2(t,r(t)) - 2\kappa(t)\cdot\bar{\phi}(t,r(t))]dt, \quad \bar{\phi}(0,r(0)) = 0$$

For the purposes of lattice construction the secondary variable's conditional expectation can be used to imply the local distribution for r

Note: Result can be extended to the general multi-factor case.

Induction of State Variables: Example, RS Model



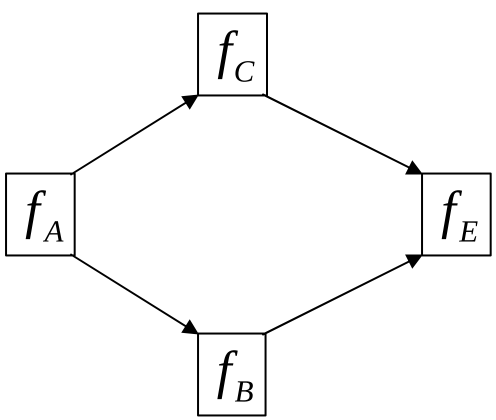
p – Local probabilities
q – Global probabilities

$$q_C = q_A \cdot p_A$$

$$q_E = q_B \cdot p_B + q_C \cdot (1 - p_C)$$

$$q_B = q_A \cdot (1 - p_A)$$

f – Conditional expectations



$$f_C = f_A + (\sigma^2 r_A^2 - 2\kappa \cdot f_A) \cdot \Delta t$$

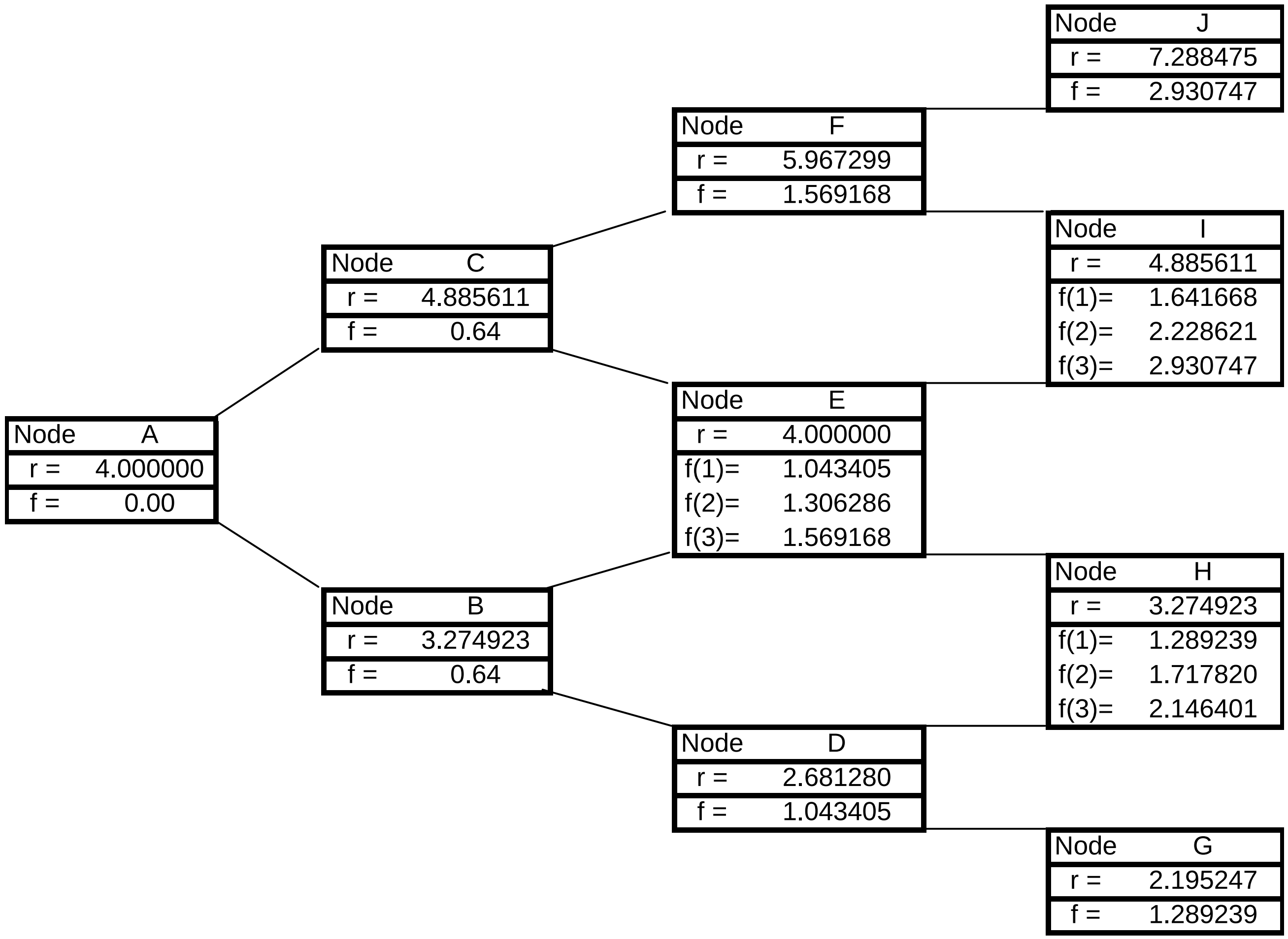
$$f_E = [f_B + (\sigma^2 r_B^2 - 2\kappa \cdot f_B) \cdot \Delta t]\frac{q_B p_B}{q_E} + [f_C + (\sigma^2 r_C^2 - 2\kappa \cdot f_C) \cdot \Delta t]\frac{q_C (1 - p_C)}{q_E}$$

$$f_B = f_A + (\sigma^2 r_A^2 - 2\kappa \cdot f_A) \cdot \Delta t$$

Simple Numerical Example (LRS Lattice)

Figure 1

The simple example of the lattice for state variables provided by LRS (1995, p. 726)*.



*** The initial term structure is set flat at 4%.The time partition for the 3-year lattice is set as one year.**
The volatility structure is given by the following parameters: κ = 0.02, σ = 0.2, γ = 1.
The lattice shows the short rate (in %) and the state variable ϕ (in % squared).
At the nodes where there are several permissible values of ϕ the minimum f(1), the maximum f(3) and their average f(2) are carried over.

Simple Numerical Example (Induction Lattice)

Figure 2
The simple example of the lattice for state variables provided by the presented induction approach*.

Node	A(0,1)
Y =	-16.09438
r =	4.000000
f =	0.00
m =	-0.10
p =	0.45
q =	1

Node	B(1,1)
Y =	-17.094379
r =	3.274923
f =	0.64
m =	-0.068089
p =	0.465956
q =	0.55

Node	C(1,2)
Y =	-15.094379
r =	4.885611
f =	0.64
m =	-0.111577
p =	0.444211
q =	0.45

Node	D(2,1)
Y =	-18.094379
r =	2.681280
f =	1.043405
m =	-0.031360
p =	0.484320
q =	0.293724

Node	E(2,2)
Y =	-16.094379
r =	4.000000
f =	1.303083
m =	-0.083711
p =	0.458144
q =	0.506381

Node	F(2,3)
Y =	-14.094379
r =	5.967299
f =	1.569168
m =	-0.119820
p =	0.440090
q =	0.199895

Node	G(3,1)
Y =	-19.094379
r =	2.195247
f =	1.289239
m =	0.011576
p =	0.505788
q =	0.151468

Node	H(3,2)
Y =	-17.094379
r =	3.274923
f =	1.685510
m =	-0.052126
p =	0.473937
q =	0.416642

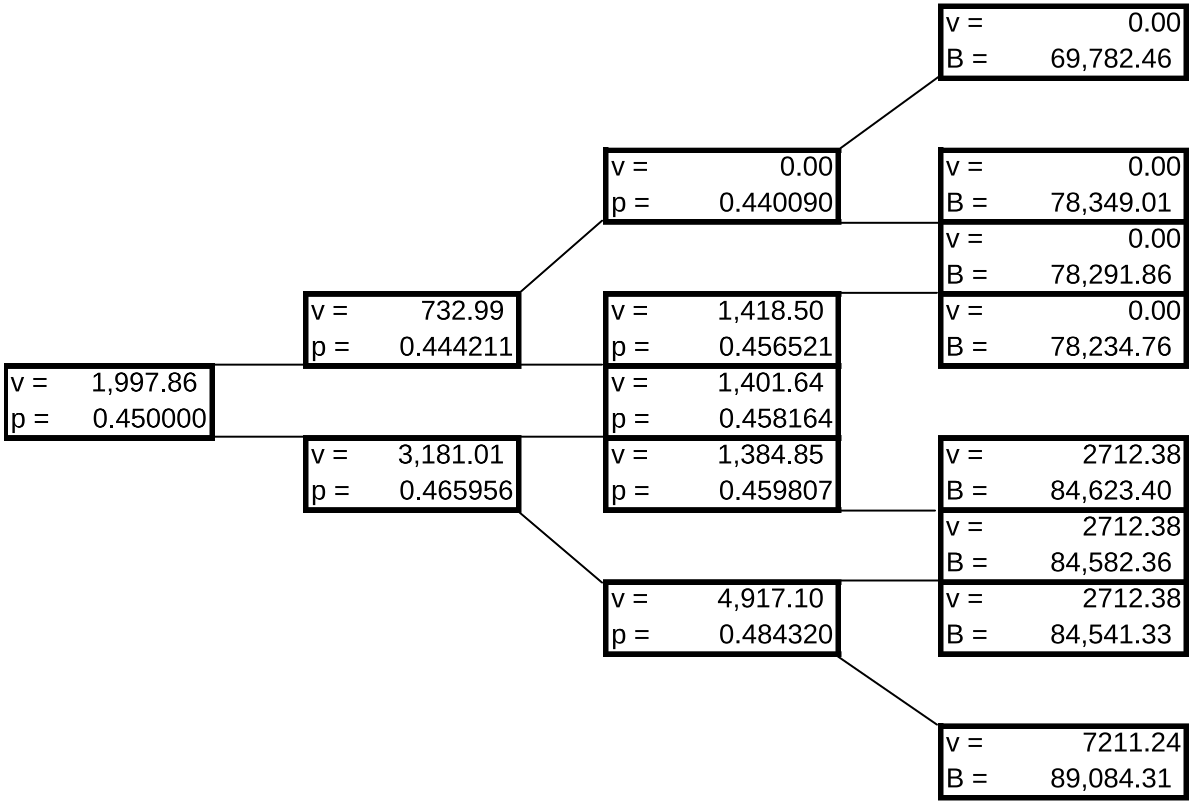
Node	I(3,3)
Y =	-15.094379
r =	4.885611
f =	2.229343
m =	-0.095312
p =	0.452344
q =	0.343919

Node	J(3,4)
Y =	-13.094379
r =	7.288475
f =	2.930747
m =	-0.125013
p =	0.437493
q =	0.087972

*** The initial term structure is set flat at 4%.The time partition for the 3-year lattice is set as one year.**
The volatility structure is given by the following parameters: κ = 0.02, σ = 0.2, γ = 1.
The lattice shows the state variable *Y*, the short rate *r* (in %), the expected value of the state variable ϕ (in % squared), the local drift *m* of *Y*, the local probability of upper jump *p*, and the unconditional node probability *q*.

Simple Numerical Example (LRS Valuation)

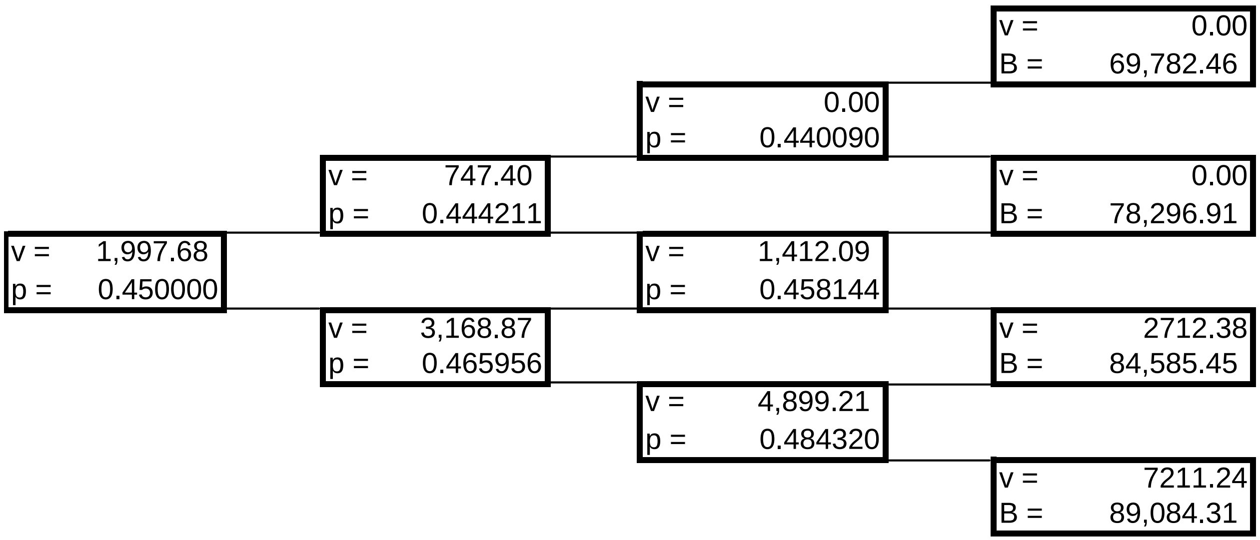
Figure 3
The simple example of the option pricing provided by LRS (1995, p. 728)*.



*** The initial term structure is set flat at 4%.The time partition for the 3-year lattice is set as one year.**
The volatility structure is given by the following parameters: κ = 0.02, σ = 0.2, γ = 1.
The 3-year option on a 8-year discount bond of 100,000 notional with strike set
at the current forward price of 81,873.07 is priced using the lattice depicted on the Figure 1.
The lattice shows the value of the option *v* and the value of the bond *B* on the terminal nodes
and the value of the option *v* and the corresponding local probabilities *p* for the remaining nodes.

Simple Numerical Example (Induction Valuation)

Figure 4
The simple example of the option pricing provided by the presented induction approach*.

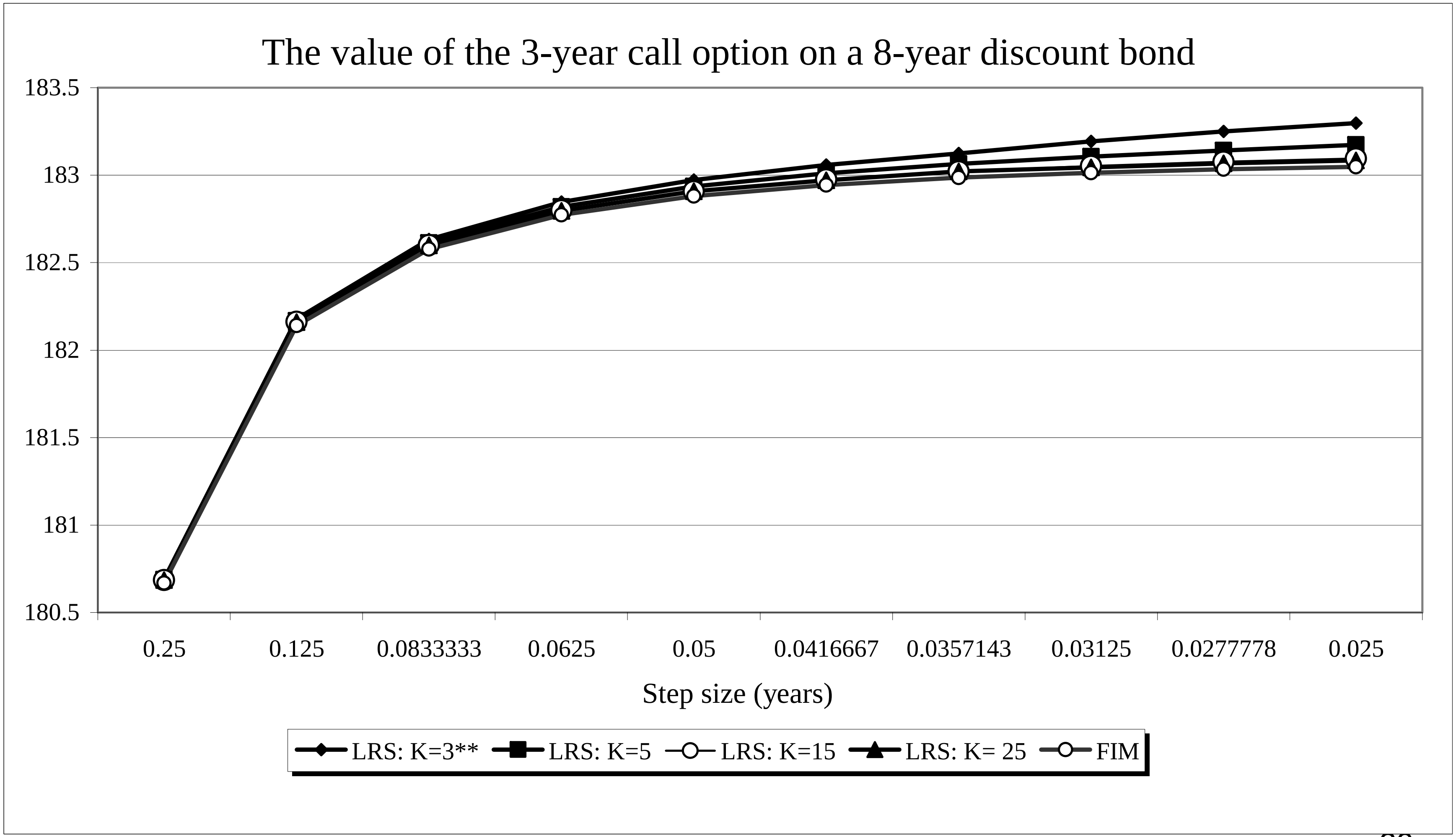


*** The initial term structure is set flat at 4%.The time partition for the 3-year lattice is set as one year.**
The volatility structure is given by the following parameters: $\kappa = 0.02$, $\sigma = 0.2$, $\gamma = 1$.
The 3-year option on a 8-year discount bond of 100,000 notional with strike set at the current forward price of 81,873.07 is priced using the lattice depicted on the Figure 2.
The lattice shows the value of the option *v* and the value of the bond *B* on the terminal nodes and the value of the option *v* and the corresponding local probabilities *p* for the remaining nodes.

Numerical Results (LRS Model)

Figure 6

The convergence of the price of the 3-year American ATM call option on a 8-year discount bond.*
The initial term structure is flat at 4% and the volatility structure is given by
the following parameters: κ =2%, σ = 20%, γ =1

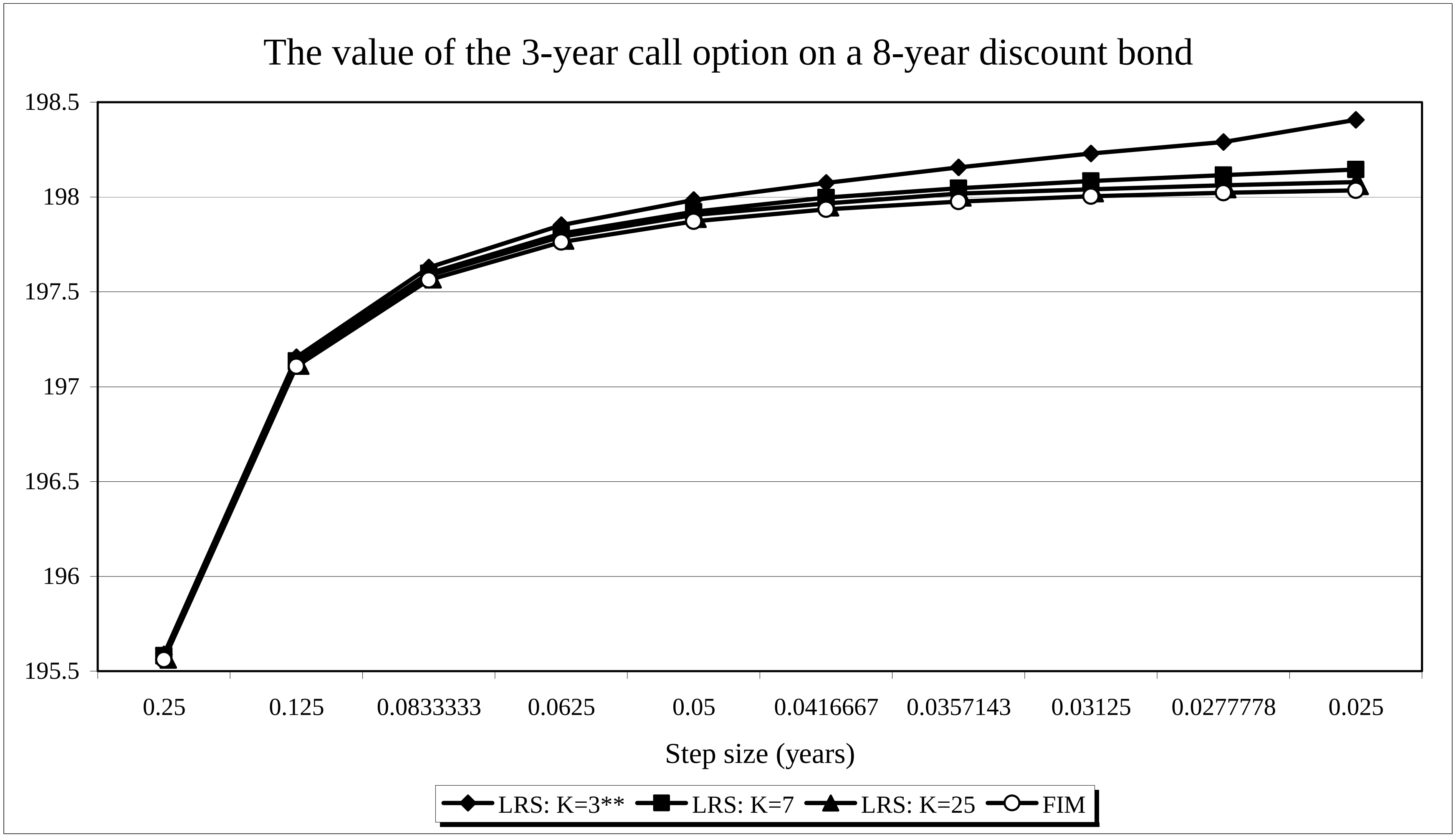


*** The notional of the bond is 10,000, and the option has strike at-the-money.**

**** K is the number of sub-nodes with different ϕ values at every node.**

Numerical Results (LRS Model)

Figure 7
The convergence of the price of an 3-year American ATM call option on a 8-year discount bond.*
The initial term structure is flat at 4% and the volatility structure is given by
the following parameters: $\kappa = 2\%$, $\sigma = 20\%$, $\gamma = 1$



*** The notional of the bond is 10,000, and the option has strike at-the-money.**

**** K is the number of sub-nodes with different ϕ values at every node.**

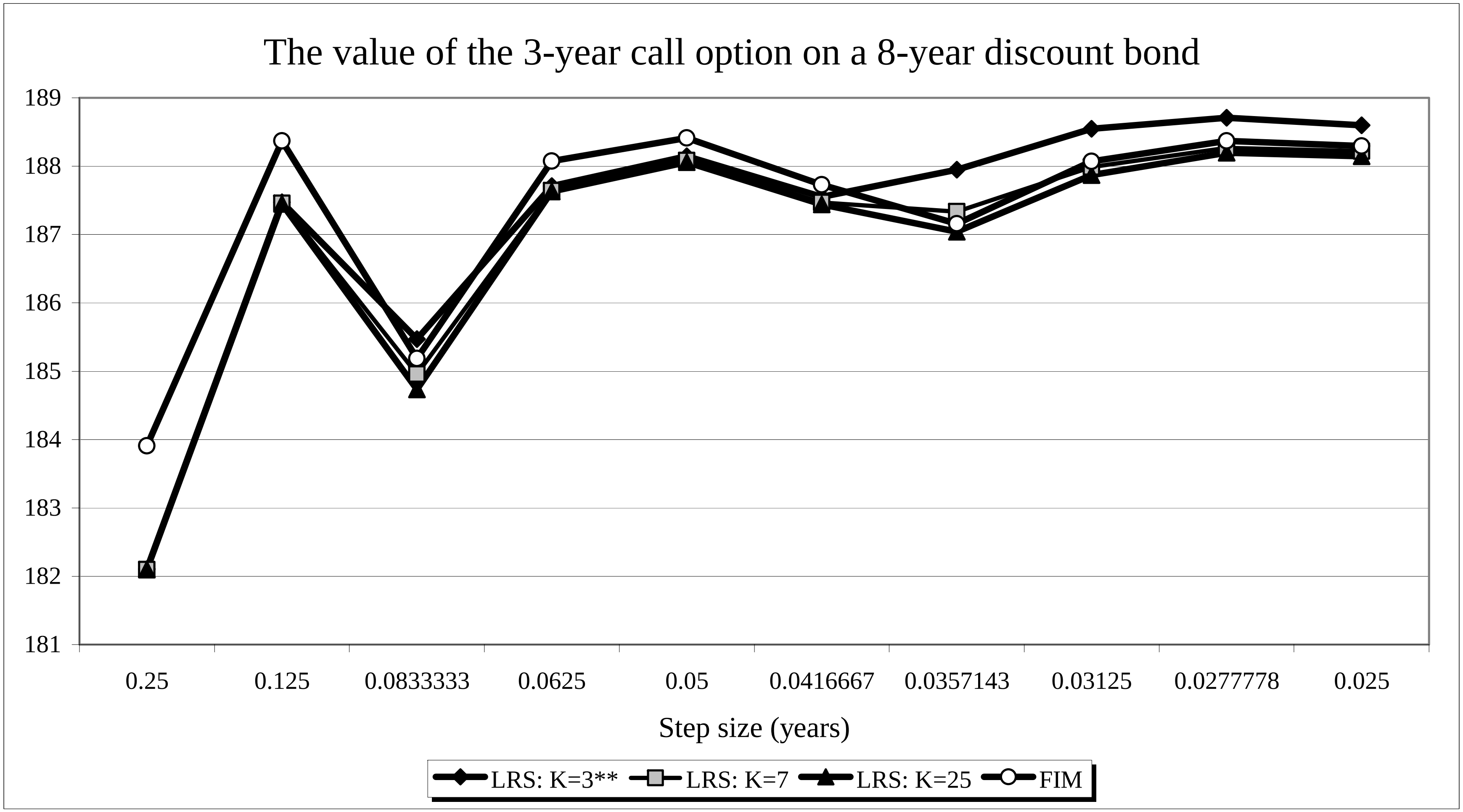
Numerical Results (LRS Model)

Figure 8

The convergence of the price of an 3-year American ATM call option on a 8-year discount bond.*

The initial term structure is given by the following discount function:* P(0,t) = exp[-(a+(b+ct)t)t]**

the following parameters: κ =2%, σ = 20%, γ =1



*** The notional of the bond is 10,000, and the option has strike at-the-money.**

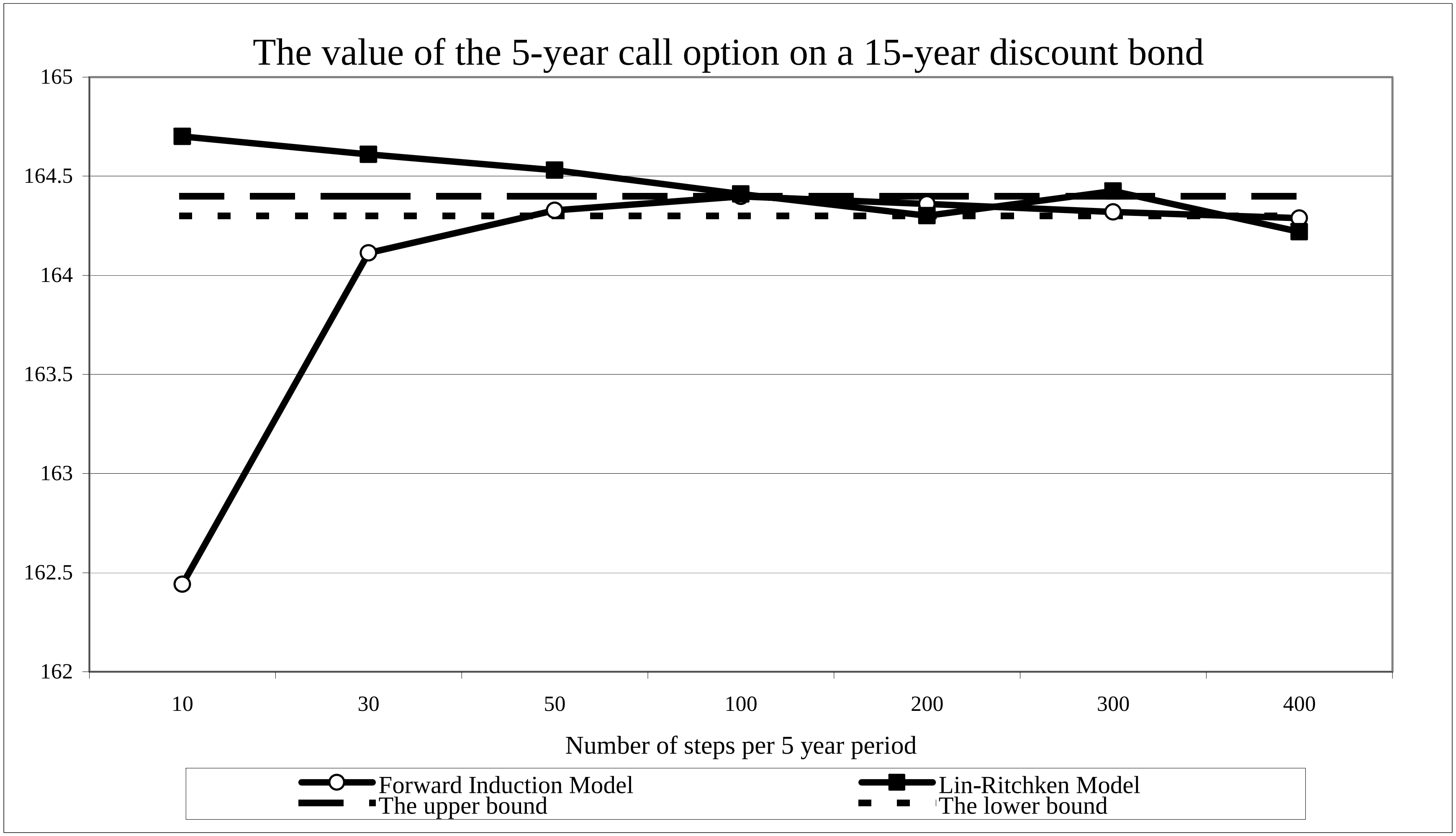
**** K is the number of sub-nodes with different ϕ values at every node.**

***** a = 4%, b = 0.2%, c = -0.004%**

Numerical Results (LR Model)

Figure 9

The convergence of the price of a 5-year European ATM call option on a 15-year discount bond.
The volatility structure is given by the following parameters: $\kappa = 1\%$, $\sigma = 2\%$, $\gamma = 0.5$
The initial term structure is flat at 6%, the notional of the bond is 10,000.



Numerical Results (LRS Model)

Table 1

The convergence of the price of a 3-year ATM American call option on a 8-year discount bond.*
The initial term structure is flat at 4% and the volatility structure is given by
the following parameters: $\kappa = 2\%, \sigma = 20\%, \gamma = 1$

The Number of steps per year	LRS: K = 3**	LRS: K = 15	LRS: K = 25	IM
4	180.692	180.686	180.686	180.668
8	182.179	182.161	182.160	182.140
12	182.633	182.600	182.598	182.577
16	182.847	182.799	182.796	182.773
20	182.973	182.909	182.909	182.880
24	183.059	182.976	182.971	182.944
28	183.124	183.021	183.021	182.986
32	183.194	183.053	183.045	183.014
36	183.250	183.077	183.067	183.034
40	183.298	183.096	183.084	183.048

*** The notional of the bond is 10,000, and the option has strike at-the-money.**
**** K is the number of sub-nodes with different f values at every node.**

Table 2

The convergence of the price of a 3-year ATM American call option on a 8-year discount bond.*
The initial term structure is flat at 4% and the volatility structure is given by
the following parameters: $\kappa = 0\%, \sigma = 20\%, \gamma = 1$

The Number of steps per year	LRS: K = 3**	LRS: K = 7	LRS: K = 25	IM
4	195.589	195.582	195.581	195.560
8	197.155	197.136	197.131	197.108
12	197.629	197.597	197.587	197.562
16	197.852	197.807	197.790	197.763
20	197.985	197.923	197.906	197.872
24	198.075	197.997	197.967	197.936
28	198.156	198.047	198.019	197.977
32	198.230	198.085	198.041	198.004
36	198.290	198.116	198.062	198.023
40	198.407	198.145	198.079	198.036

*** The notional of the bond is 10,000, and the option has strike at-the-money.**
**** K is the number of sub-nodes with different f values at every node.**

Table 3

The convergence of the price of a 3-year ATM American call option on a 8-year discount bond.*
The initial term structure is given by the following discount function:
and the volatility structure is: $\kappa = 0\%, \sigma = 20\%, \gamma = 1$, $P(0,t) = \exp[-(a+(b+ct)t)t]$

The Number of steps per year	LRS: K = 3**	LRS: K = 7	LRS: K = 25	IM
4	182.105	182.099	182.098	183.907
8	187.475	187.456	187.451	188.372
12	185.472	184.960	184.733	185.187
16	187.708	187.643	187.629	188.074
20	188.147	188.083	188.059	188.412
24	187.551	187.470	187.440	187.729
28	187.951	187.334	187.041	187.160
32	188.548	187.990	187.872	188.072
36	188.708	188.270	188.198	188.370
40	188.599	188.222	188.148	188.296

*** The notional of the bond is 10,000, and the option has strike at-the-money.**
**** K is the number of sub-nodes with different f values at every node.**

Numerical Results (LR and RC Models)

Table 4
The convergence of the price of a 5-year ATM European call option on a 15-year discount bond.
The volatility structure is given by the following parameters: $\kappa = 1\%$, $\sigma = 2\%$, $\gamma = 0.5$
The initial term structure is flat at 6%, the notional of the bond is 10,000.

The number of time steps for 5 year period	10	30	50	100	200	300	400
Induction Method	162.442	164.113	164.327	164.398	164.360	164.319	164.289
Lin-Ritchken Method	164.700	164.610	164.530	164.410	164.300	164.425	164.220

Table 5
The convergence of the price of a 0.5-year ATM European call option on a 2-year discount bond with 10,000 notional.
The initial term structure is given by the initial instantaneous forward rate: f(0,t) = 0.07 - 0.02 exp(- 0.18t)
and the volatility structure is: κ **= 10%, a = 2%, b = 0.3%, c =0**

The number of time steps per semiannual period	2	3	4	5	10	25	50	100	200	500	1000
Induction Method	72.53	87.68	76.39	84.70	78.85	81.18	80.12	80.26	80.31	80.34	80.34
Ritchken-Chuang Method (Numerical)	72.56	87.68	76.49	84.70	78.92	81.18	80.14	80.26	80.32	80.34	80.34
Ritchken-Chuang Method (Analytical)	80.33	80.33	80.33	80.33	80.33	80.33	80.33	80.33	80.33	80.33	80.33

Table 6
The convergence of the price of a 0.5-year ATM European call option on a 2-year discount bond with 10,000 notional.
The initial term structure is given by the initial instantaneous forward rate: f(0,t) = 0.07 - 0.02 exp(- 0.18t)
and the volatility structure is: κ **= 10%, a = 2%, b = 0.3%, c = 0.25%**

The number of time steps per semiannual period	2	3	4	5	10	25	50	100	200	500	1000
Induction Method	79.25	96.10	83.95	93.14	86.95	89.62	88.50	88.67	88.74	88.77	88.77
Ritchken-Chuang Method (Numerical)	79.25	96.11	83.95	93.14	86.98	89.62	88.52	88.68	88.74	88.77	88.77
Ritchken-Chuang Method (Analytical)	88.76	88.76	88.76	88.76	88.76	88.76	88.76	88.76	88.76	88.76	88.76

Lattice Induction Approach, Multi-Factor HJM Model

- **Apply Induction Approach to every factor**
- **Combine into Multi-factor lattice:**

  Hull&White (1994):

  Combination of two one-factor models into two-factor model

  Trinomial model for every factor – Nine-nomial for two-factor model

  Efficient two-factor lattice, Feasible three-factor lattice

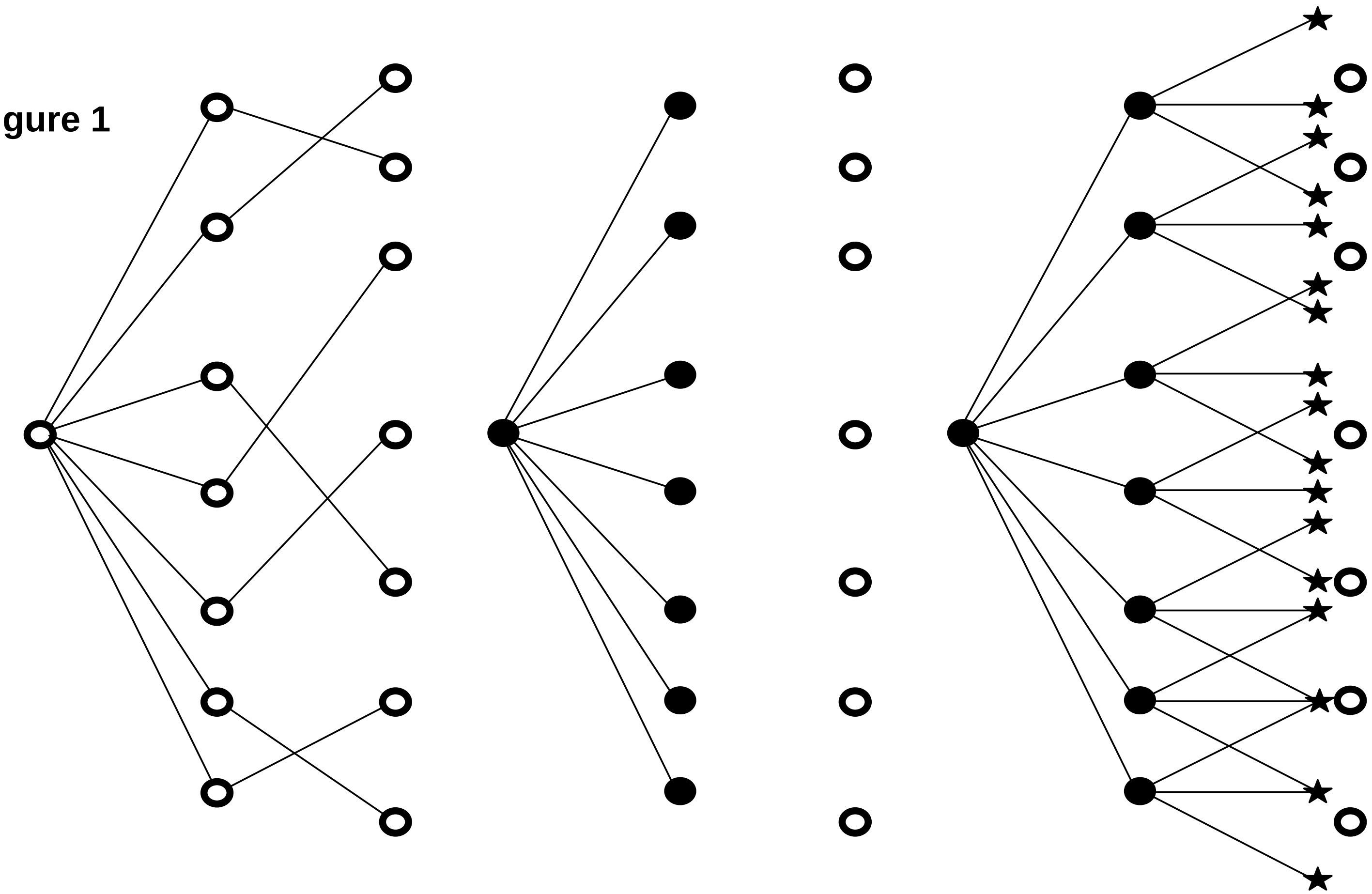
Lattice Evolution Approach, Multi-Factor HJM Model

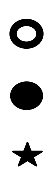
- **Select two-three most important derivative variables:**
  short rate, zero-coupon bond prices for various maturities,
  forward rates, swap rates, CMS spread etc.
- **Set up “important variable space” (IVS)**
- **Evolve a non-recombining lattice discretely and bundle cells in IVS**
- **Reconnect cells into the recombining lattice**
- **Compute primary state variables X of resulting cells (Induction Approach)**
- **Adjust probabilities to match moments for primary state variable (s)**
- **Compute conditional expectation of secondary state variables Y**

Efficient lattice: finite number of factors and a few important variables

Lattice Evolution Approach Based on Simulation I

Figure 1



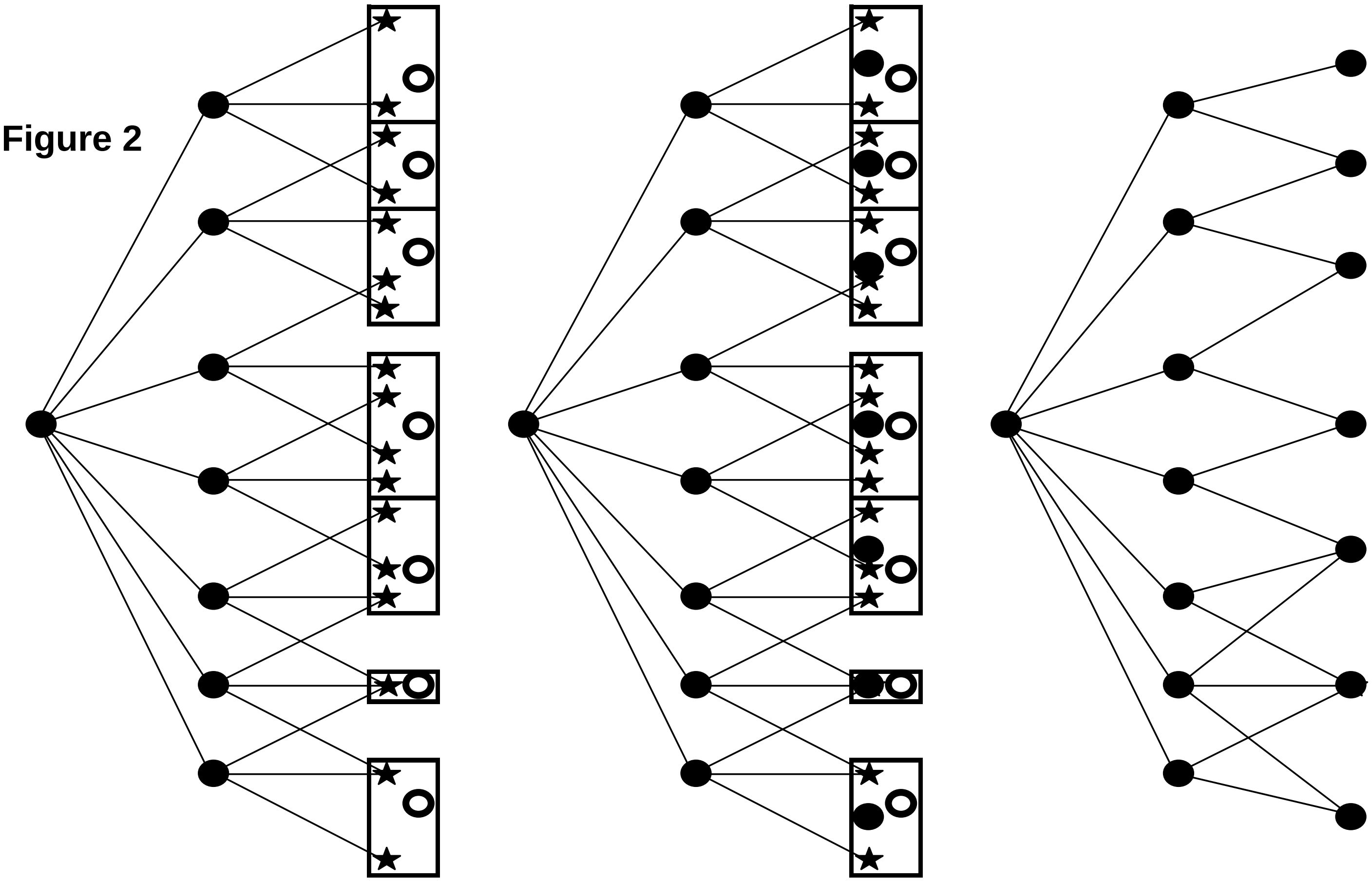


- Simulation Points
- Lattice Points
- Evolution Split Points

- **MC Simulation**
- **First Lattice Layer – from MC Simulation**
- **Lattice Evolution**

Lattice Evolution Approach Based on Simulation II

Figure 2

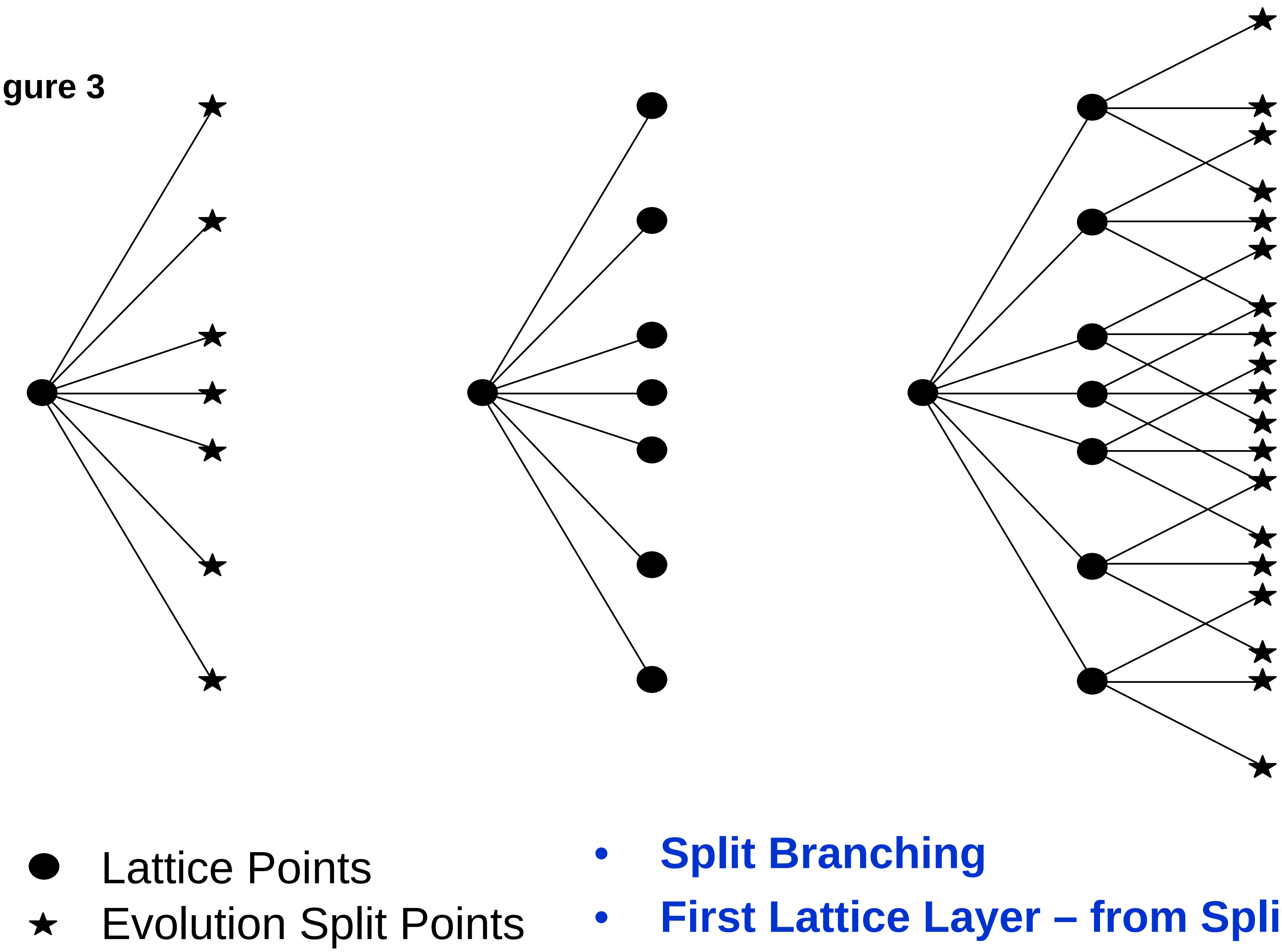


- ○ Simulation Points
- ● Lattice Points
- ★ Evolution Split Points

- **Split States Redistribution (Bundling)**
- **Lattice States Computation using Induction**
- **Lattice Building: Reconnection + Probabilities**

Lattice Evolution Approach Based on Space Clustering I

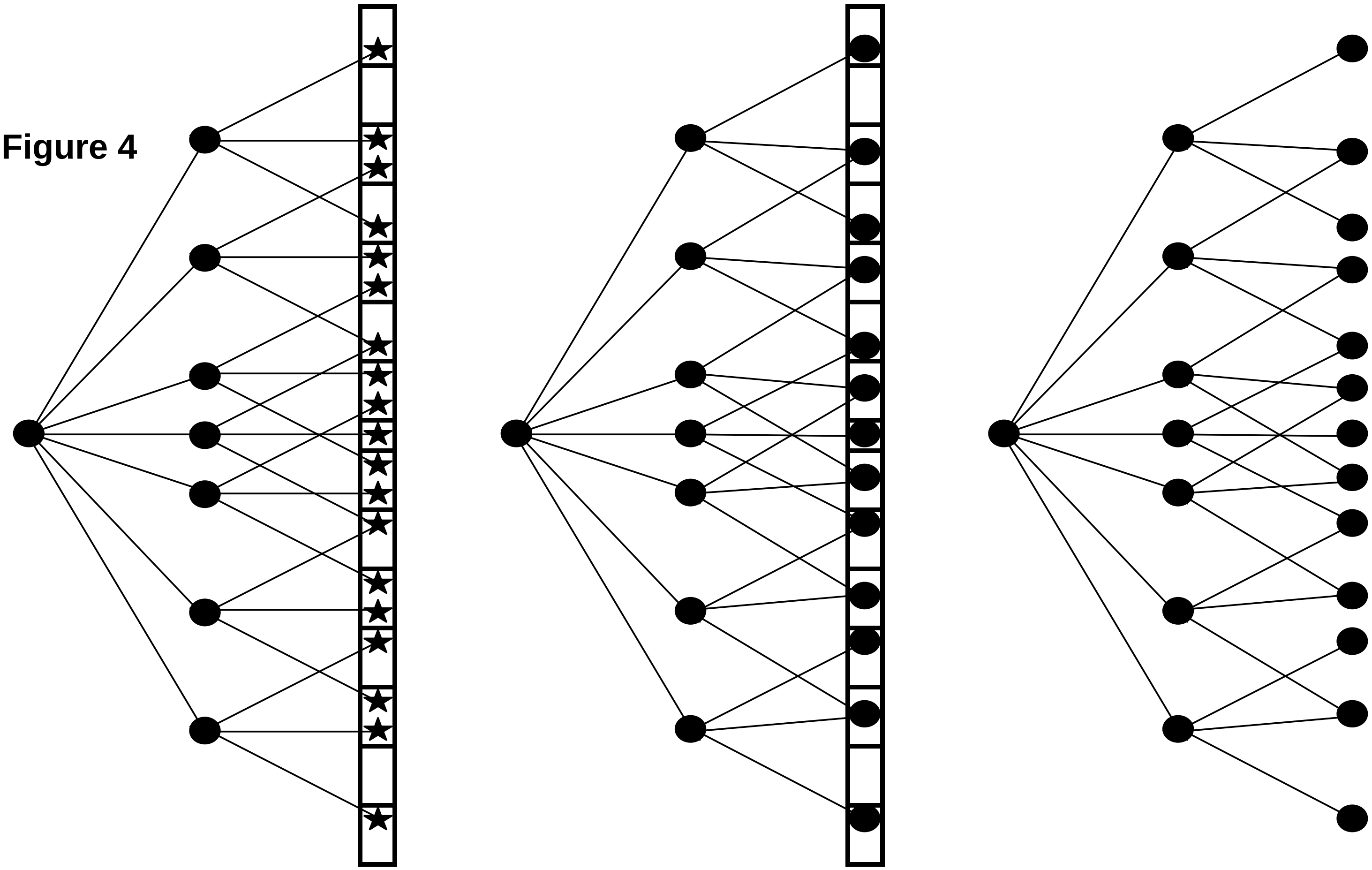
Figure 3



- **Split Branching**
- **First Lattice Layer – from Split Branching**
- **Lattice Evolution**

Lattice Evolution Approach Based on Space Clustering II

Figure 4



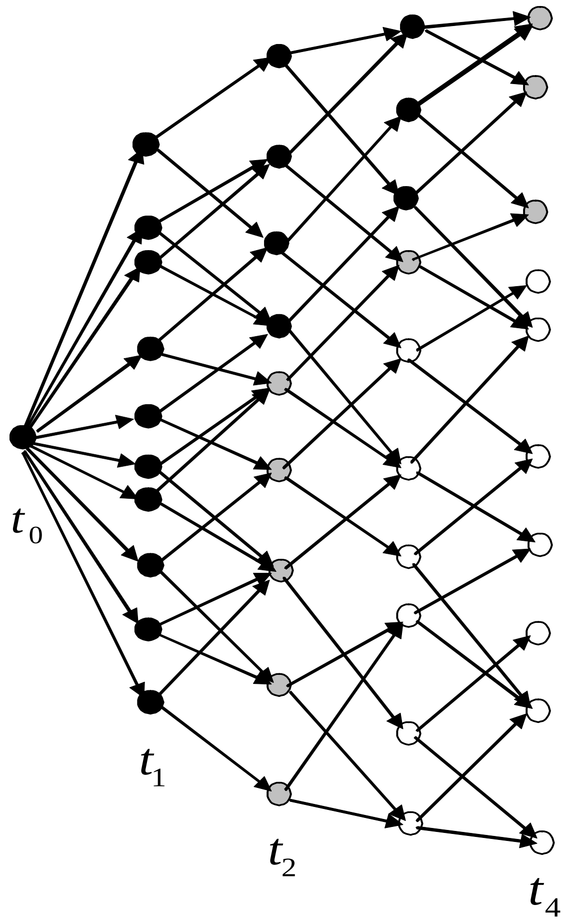
- Lattice Points
- ★ Evolution Split Points

- **Split States Redistribution (Bundling)**
- **Lattice States Computation using Induction**
- **Lattice Building: Reconnection + Probabilities**

Mapping Exercise Boundary on a given MC Path

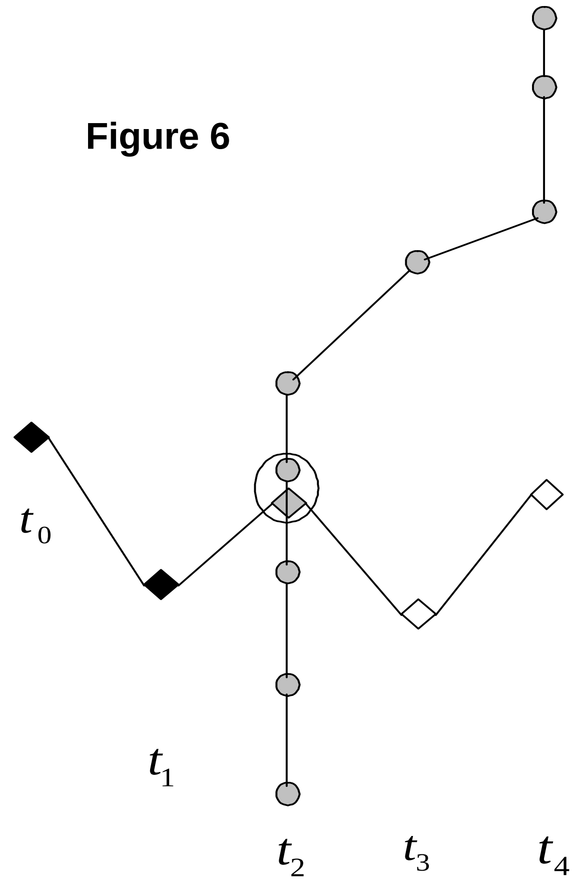
Exercise Boundary
Bermudan Call Option
Lockout at t2, Maturity at t4

Figure 5



Exercise Boundary Mapping
On a given MC Path

Figure 6



Exercise is not optimal or possible
Exercise is optimal
Option does not exist

Exercise is not optimal or possible
Exercise is optimal
Option does not exist

Numerical Examples: One- and Two- factor Two-term Models

One-factor model $$\sigma(t,T) = (\sigma_1 e^{-\beta_1(T-t)} + \sigma_2 e^{-\beta_2(T-t)}) \cdot r^{1-\rho} \cdot (1 + ae^{-bt}) \cdot r_0^{\rho-1}$$

Parameters	σ_1	σ_2	β_1	β_2	a	b	r_0	ρ
Values	0.1497	-0.3146	0.0548	2.9452	1.7302	0.2693	0.0675	0.75

Two-factor model $$\sigma_j(t,T) = (\sigma_{1j} e^{-\beta_{1j}(T-t)} + \sigma_{2j} e^{-\beta_{2j}(T-t)}) \cdot (r / r_0)^{1-\rho}$$

$$j = 1,2$$

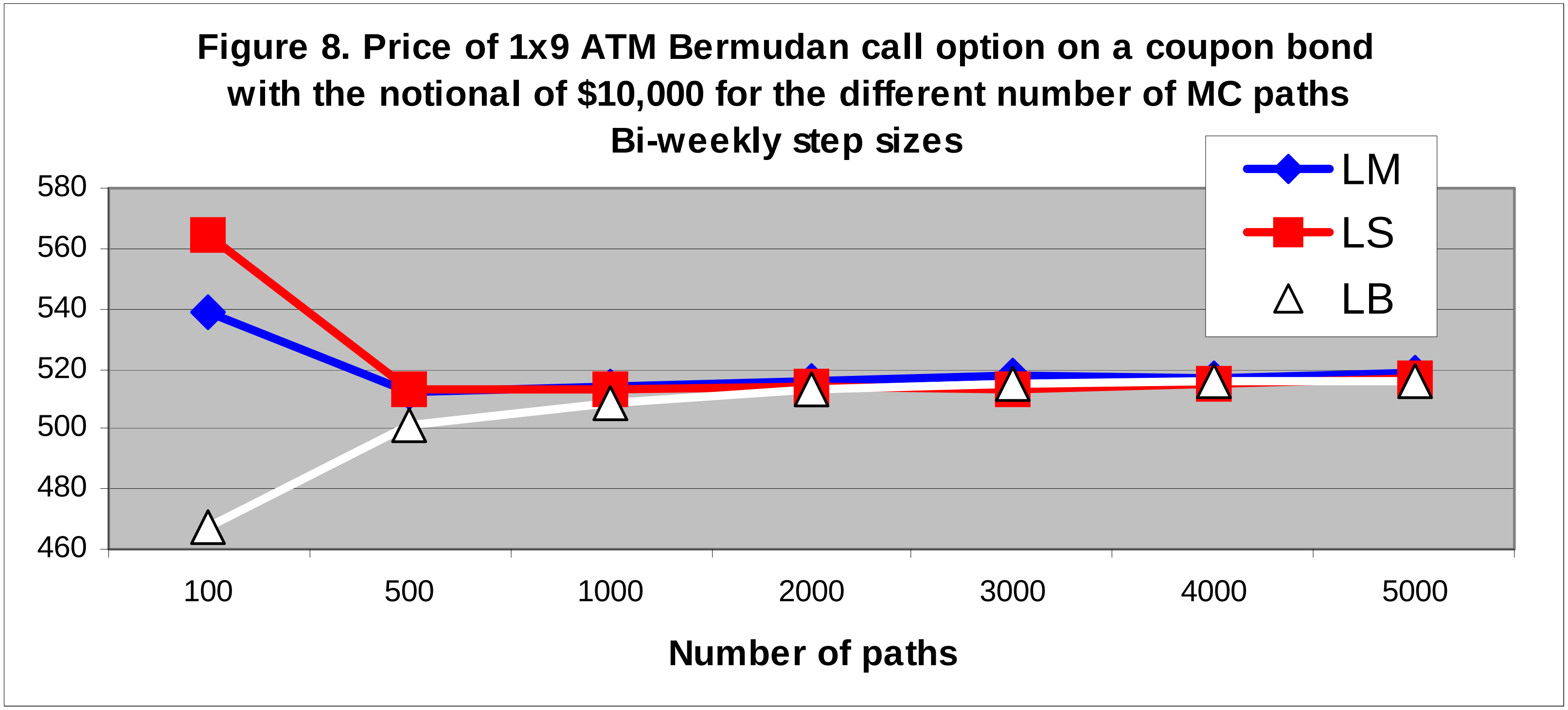
Parameters	σ_{11}	σ_{21}	σ_{12}	σ_{22}	β_{11}	β_{21}	β_{12}	β_{22}	r_0	ρ
Values	0.1926	-0.2074	-0.1778	0.0148	0.05	2.6	3	-0.05	0.0675	0.75

LB – lattice-based approach

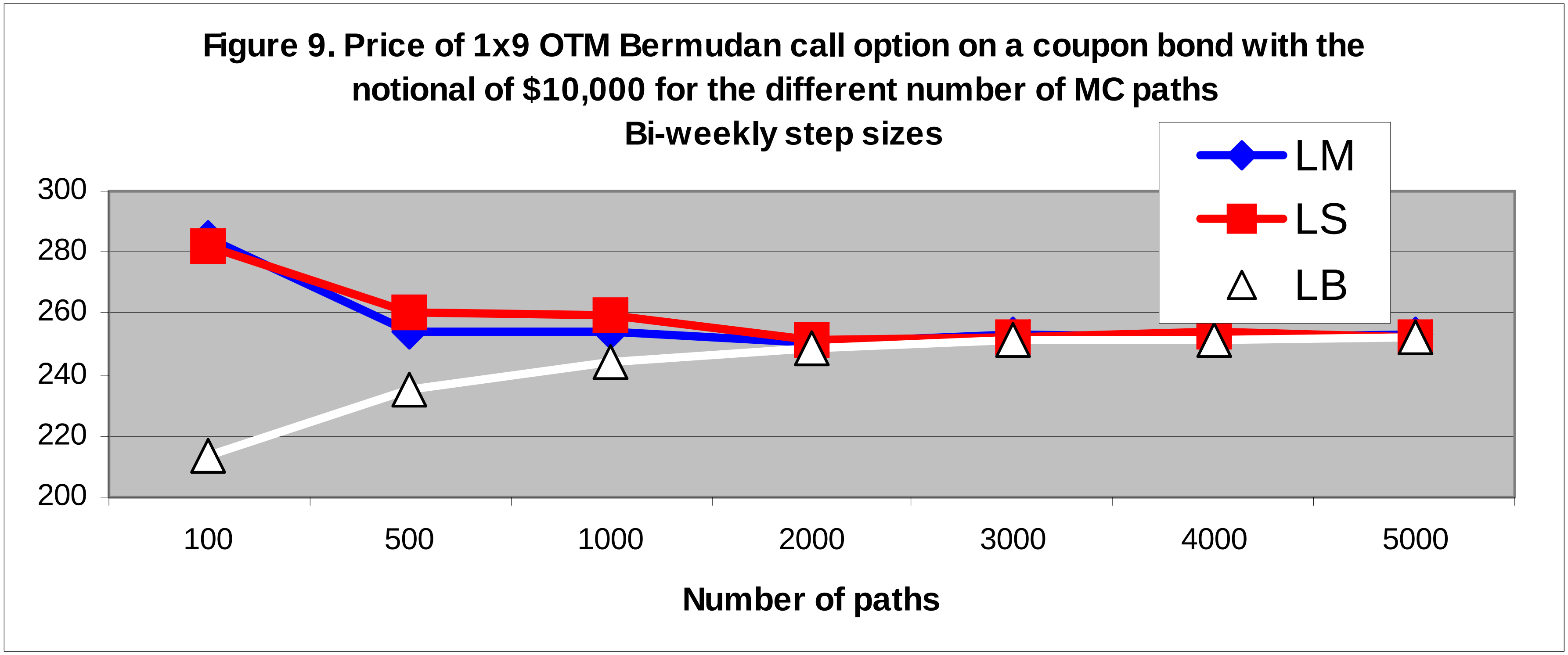
LM – lattice-MC approach: Mapping Lattice Exercise Boundary to MC set

LS – Least Square (regression-based) approach, LS(2001)

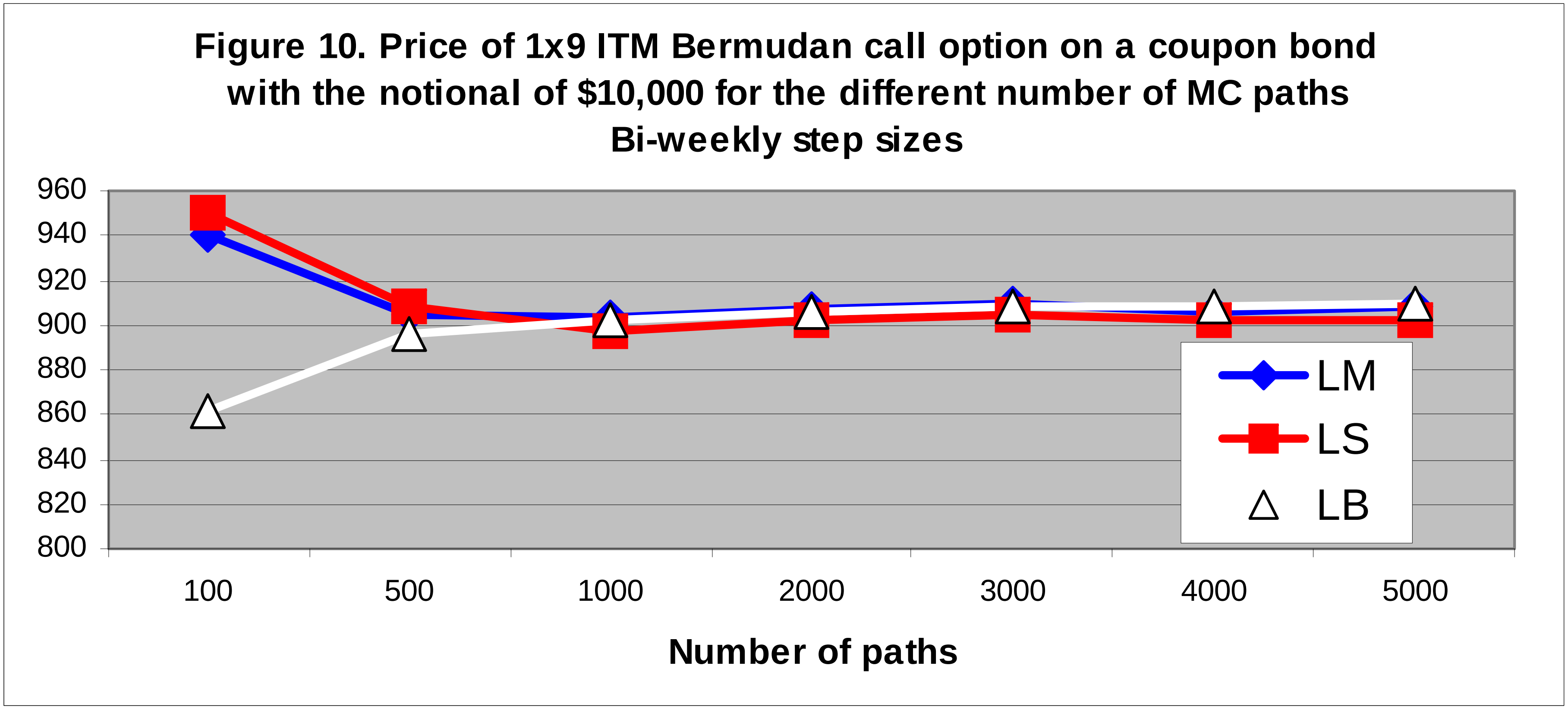
Numerical Examples: Convergence Properties of LB, LM, and LS methods
One-factor Model



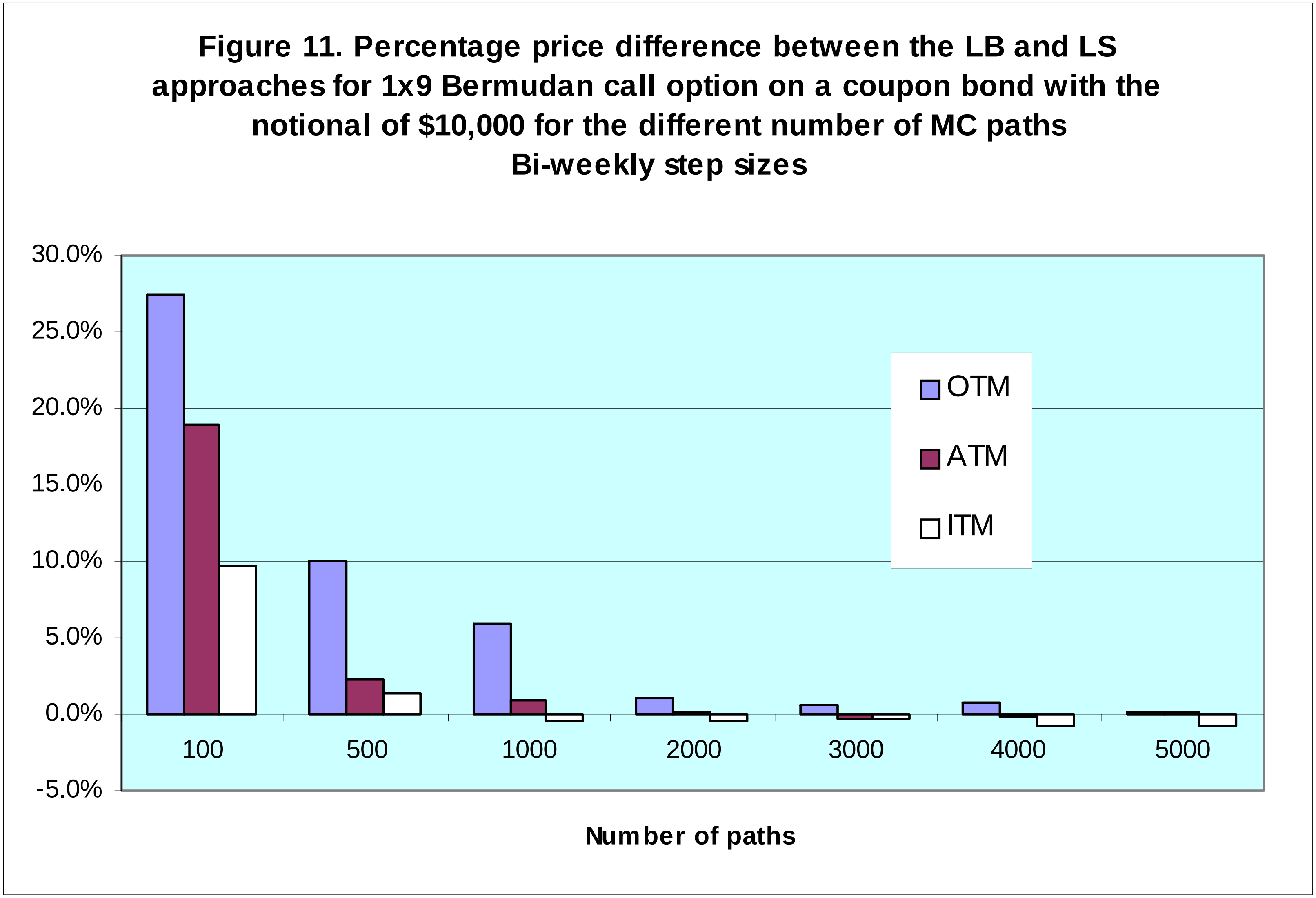
Numerical Examples: Convergence Properties of LB, LM, and LS methods
One-factor Model



Numerical Examples: Convergence Properties of LB, LM, and LS methods
One-factor Model



Numerical Examples: Comparison of LB and LS methods
One-factor Model



Numerical Examples: Comparison of LB and LS methods
Two-factor Model

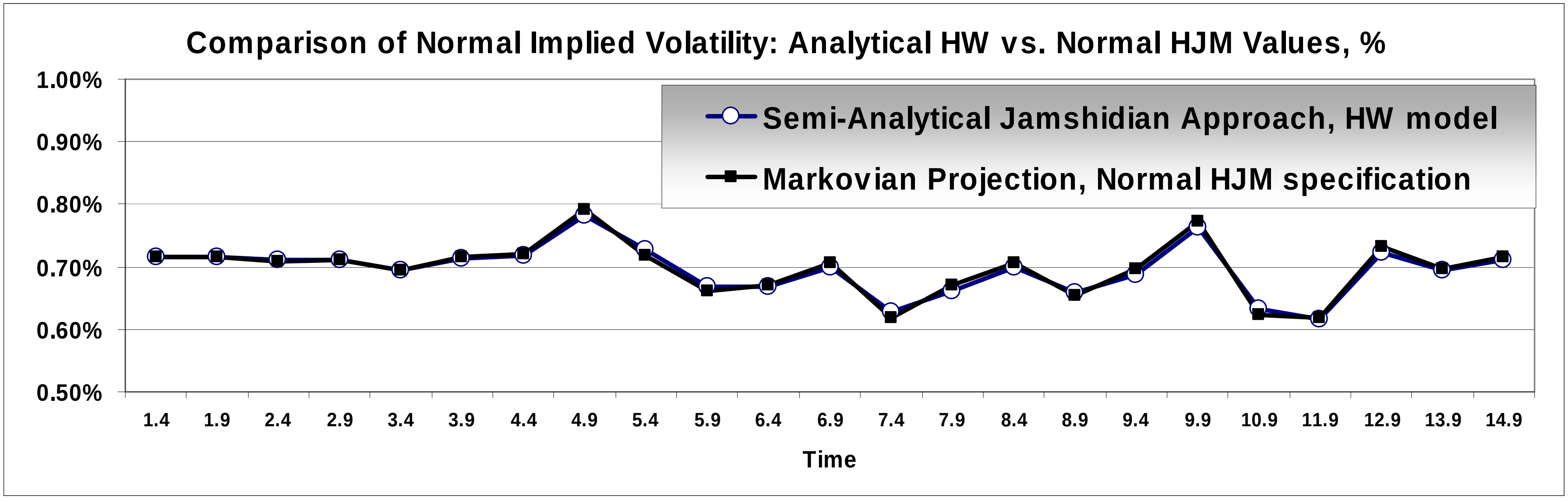
Percentage price difference between the LB and LS approaches for 1x1 American call option on a coupon bond with the notional of $10,000 for the different number of MC paths (Bi-weekly step sizes)

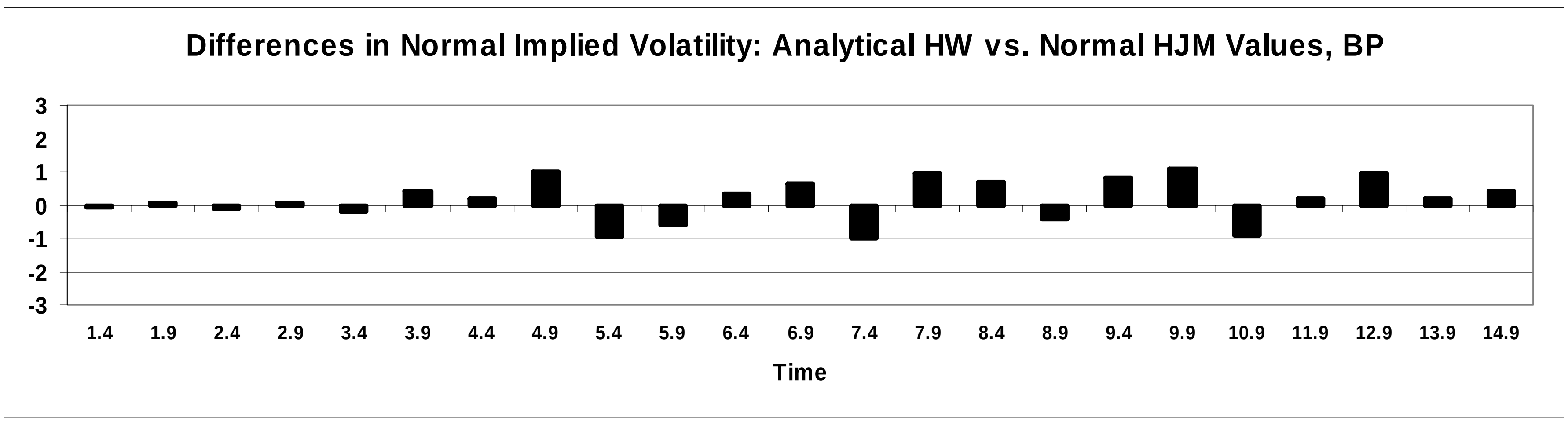
	LS	LB	% Difference
OTM	6.75731	6.75594	0.02%
ATM	35.7921	35.8909	-0.28%
ITM	99.7744	99.6378	0.14%

Example: One-Factor Two-State Variable HJM Model Pricing Bermudan Swaption on Range Accrual Swap

Specification of Bermudan Swaption on Range Accrual Swap	
Product	Bermudan Swaption
Underlying Swap	Range Accrual Swap
Exercise Structure	15 NC 1, Semiannual (SA)
Swap Fixed Exotic Coupon	8.6%
Swap Funding Rate	LIBOR 6M - 38 BP
Lower Range	0%
Upper Range	7%
Notional	10,000

Pricing Bermudan Swaption on Range Accrual Swap
Application of Markovian Projection: Consistency Test

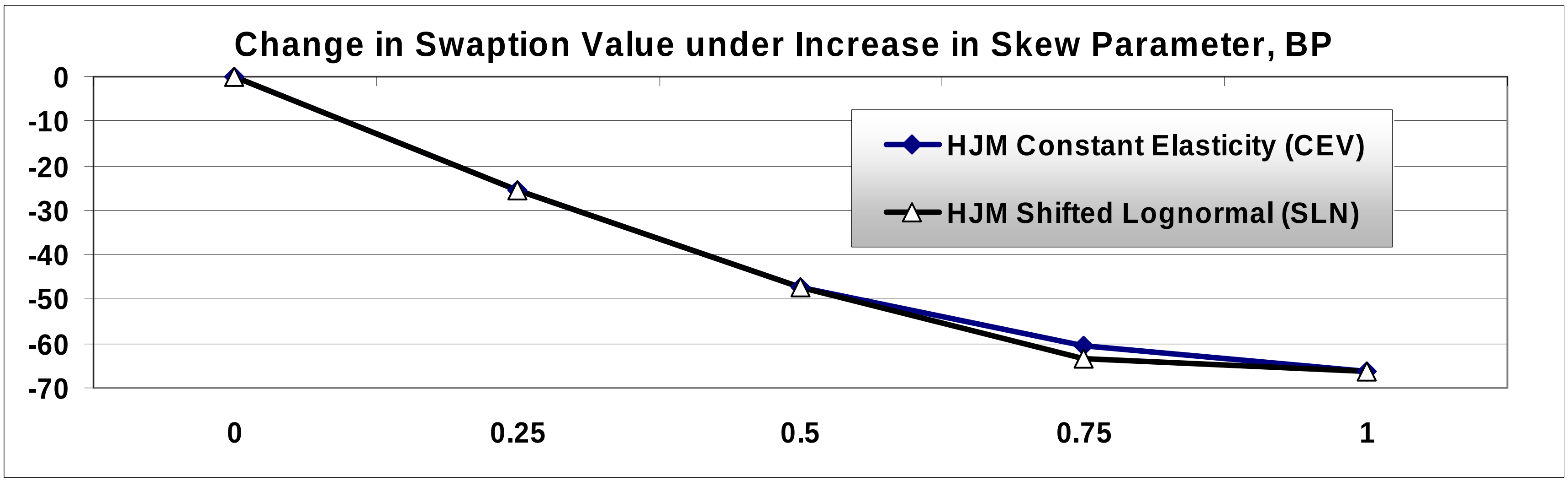




Pricing Bermudan Swaption on Range Accrual Swap Application of Lattice Evolution Approach (Space Clustering) and Hagan (2002) Adjuster Approach

HJM model prices, I factor					
Flat Skew	0	0.25	0.5	0.75	1
HJM Constant Elasticity (CEV)	2,248	2,223	2201	2,187	2,182
HJM Shifted Lognormal (SLN)	2,248	2,223	2201	2,185	2,182
Change Due to CEV Skew, BP	-	-26	-47	-61	-66
Change Due to SLN Skew, BP	-	-26	-47	-63	-66

HW Model Price: 2,246



Zero and unit skew values correspond to Normal and Lognormal specifications respectively

Advantages of Lattice Induction/Evolution Approach

- **Simple forward induction for state variables;**
- **No numerical interpolations and iterative procedures;**
- **Lower dimensionality;**
- **No adjustments and no significant loss of information;**
- **Simple Backward Induction for interest rate derivatives pricing**
- **Parsimonious and computationally less expensive alternative to existing methods.**

CONCLUSIONS

Multi-factor Markovian HJM framework.

Parsimonious multi-factor volatility specification.

Application of Markovian projection for calibration.

Generic lattice-building approach for pricing American-type interest rate derivatives.

Numerical Results:

Lattice Induction/Evolution Approach - Parsimonious and Efficient model for pricing exotic interest rate derivatives

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